Math 2233 Summer 2025
Multivariable Calculus
Mastery Quiz 8
Due Monday, July 28

Sorry, I know I posted this late. I'll still accept it on Tuesday, but if you can get it in on Monday we'll both find the rest of the week easier.

This week's mastery quiz has three topics. Everyone should submit S4, unless you nailed it on the midterm. (Check Blackboard!) If you have a 4/4 on M3 or M4, (meaning you have gotten 2/2 twice), you don't need to do them again.

Don't worry if you make a minor error, but try to demonstrate your mastery of the underlying material. Feel free to consult your notes, but please **don't discuss the actual quiz questions with other students in the course**.

Remember that you are trying to demonstrate that you understand the concepts involved. For all these problems, justify your answers and show your work. Do not just write "yes" or "no" or give a single number.

Please turn this quiz in class on Wednesday. You may print this document out and write on it, or you may submit your work on separate paper; in either case make sure your name and recitation section are clearly on it. If you absolutely cannot turn it in in person, you can submit it electronically but this should be a last resort.

Topics on This Quiz

- Major Topic 3: Optimization
- Major Topic 4: Integration
- Secondary Topic 4: Integral Applications

Name:

Name:

M3: Optimization

(a) Find and classify the critical points of $f(x, y) = 2x^3 - 6xy + y^2$.

Solution: We have

$$f_x(x, y) = 6x^2 - 6y$$
$$f_y(x, y) = -6x + 2y$$

The second equation tells us $y = 3x$. Substituting that into the first equation gives $6x^2 - 18x = 0$ and thus either $x = 0$ or $x = 3$. So our two critical points are $(0, 0)$ and $(3, 9)$.

We have

$$\begin{array}{lll} f_{xx}(x, y) = 12x & f_{xx}(0, 0) = 0 & f_{xx}(3, 9) = 36 \\ f_{xy}(x, y) = -6 & f_{xy}(0, 0) = -6 & f_{xy}(3, 9) = -6 \\ f_{yy}(x, y) = 2 & f_{yy}(0, 0) = 2 & f_{yy}(3, 9) = 2 \end{array}$$

Then for $(0, 0)$ we have $D = 0 \cdot 2 - (-6)^2 = -36 < 0$, so this is a saddle point.

For $(3, 9)$ we have $D = 36 \cdot 2 - (-6)^2 = 36 > 0$, and $f_{xx} = 36 > 0$, so this is a local minimum.

(b) Find the maximum and minimum values of $g(x, y, z) = y^2 - 10z$ subject to the constraint $x^2 + y^2 + z^2 = 36$.

Solution: We get the three equations

$$0 = \lambda \cdot 2x$$
$$2y = \lambda 2y$$
$$-10 = \lambda 2z$$

The first equation tells us that $x = 0$ *or* that $\lambda = 0$. Note that either of these things can happen! We could analyze both these cases, but I want to see if some equation is more useful than the other.

The second equation, similarly, tells us that $\lambda = 1$ *or* that $y = 0$. Again, we can analyze either case.

The third equation tells us that $z = -5/\lambda$. In particular, we see that neither z nor λ can be zero.

So now we can go back to the first equation; we know $\lambda \neq 0$ and thus $x = 0$. The second equation genuinely gives us two possibilities.

If $\lambda = 1$, then $z = -5$; from the constraint equation we have $y^2 + 25 = 36$ so $y = \pm\sqrt{11}$. Our two critical poitns are $(0, \sqrt{11}, -5)$ and $(0, -\sqrt{11}, -5)$. We compute

$$g(0, \sqrt{11}, -5) = 11 + 50 = 61$$
$$g(0, -\sqrt{11}, -5) = 11 + 50 = 61.$$

(If you hadn't noticed the other two critical points, you should realize now that something has gone wrong: the function isn't constant but you've only found one value.)

If $y = 0$ then the constraint equation gives us $z^2 = 36$ so $z = \pm 6$. Our critical points here are $(0, 0, 6)$ and $(0, 0, -6)$ and so we compute

$$g(0,0,6) = -60$$
$$g(0,0,-6) = 60.$$

Thus the globam maximum is 61, achieved at $(0, \sqrt{11}, -5)$ and $(0, -\sqrt{11}, -5)$; the global minimum is -60, achieved at $(0, 0, 6)$.

(c) Find (but don't classify) the critical points of $g(x, y, z) = x^2+y^2+3z^2-2x-8y-z^3+5$.

Solution: We have

$$g_x(x,y,z) = 2x - 2$$
$$g_y(x,y,z) = 2y - 8$$
$$g_z(x,y,z) = 6z - 3z^2$$

The first equation says $x = 1$, and the second says $y = 4$. Then the third gives us $3z(2 - z) = 0$, and so $z = 0$ or $z = 2$. So there are two critical points: $(1, 4, 0)$ and $(1, 4, 2)$.

M4: Integration

(a) We want to integrate the function $f(x, y, z) = (x^2+y^2+z^2)^{3/2}$, over the region enclosed by the cone $z = \sqrt{3x^2 + 3y^2}$ and the sphere $x^2 + y^2 + z^2 = 16$. Set up three different iterated integrals to compute this, in cartesian, cylindrical, and spherical coordinates.

Choose one of the integrals you set up and evaluate it.

Solution: The two surfaces intersect when $x^2 + y^2 + 3x^2 + 3y^2 = 16$, so $x^2 + y^2 = 4$ and we have a circle of radius 2. Thus we get

$$\int_{-2}^{2} \int_{-\sqrt{4-x^2}}^{\sqrt{4-x^2}} \int_{\sqrt{3x^2+3y^2}}^{\sqrt{16-x^2-y^2}} (x^2 + y^2 + z^2)^{3/2} \, dz \, dy \, dx.$$

I do not want to do this integral.

Cylindrical looks a bit better. We worked out above that the radius of the circle of intersection is 2; and theta goes all the way around, from 0 to 2π. Since $x^2 + y^2 = r^2$ we get a simpler function, and not forgetting the Jacobian we have

$$\int_{0}^{2} \int_{0}^{2\pi} \int_{\sqrt{3}r}^{\sqrt{16-r^2}} (r^2 + z^2)^{3/2} \cdot r \, dz \, d\theta \, dr.$$

But I still don't want to do that.

In spherical things look pretty okay! We have θ going all the way around from 0 to 2π, and in each direction ρ varies from 0 to 4. We just need to figure out the bounds on ϕ, which is the interior angle of the cone. Since the line has slope $\sqrt{3}$ we can compute this as $\arctan(1/\sqrt{3}) = \pi/6$. Or we could just recognize that this is a 30-60-90 triangle, which gives an interior angle of 30 degrees. Even better, our function is just ρ^3. Thus we have

$$\int_0^4 \int_0^{2\pi} \int_0^{\pi/6} \rho^3 \cdot \rho^2 \sin\phi \, d\phi \, d\theta \, d\rho$$

So we compute the spherical integral:

$$\begin{aligned}
\int_0^4 \int_0^{2\pi} \int_0^{\pi/6} \rho^3 \cdot \rho^2 \sin\phi \, d\phi \, d\theta \, d\rho &= \int_0^4 \int_0^{2\pi} \rho^5(-\cos\phi)\Big|_0^{\pi/6} \, d\theta \, d\rho \\
&= \int_0^4 \int_0^{2\pi} \rho^5(1-\sqrt{3}/2) \, d\theta \, d\rho \\
&= (1-\sqrt{3}/2) \int_0^4 \rho^5\theta\Big|_0^{2\pi} \, d\rho \\
&= 2\pi(1-\sqrt{3}/2) \int_0^4 \rho^5 \, d\rho \\
&= \pi(2-\sqrt{3}) \frac{\rho^6}{6}\Big|_0^4 \\
&= \pi(2-\sqrt{3})4^6/6 = 2048\pi(2-\sqrt{3})/3.
\end{aligned}$$

(b) Use the change of variables $s = y, t = y - x^2$ to evaluate $\iint_R x \, dA$ over the region in the first quadrant bounded by $y = 0, y = 36, y = x^2, y = x^2 - 1$.

Solution: We have $y = s$ and $x = \sqrt{y-t} = \sqrt{s-t}$, so we compute

$$\left|\frac{\partial(x,y)}{\partial(s,t)}\right| = \begin{vmatrix} \frac{1}{2\sqrt{s-t}} & \frac{-1}{2\sqrt{s-t}} \\ 1 & 0 \end{vmatrix} = \left|\frac{1}{2\sqrt{s-t}}\right| = \frac{1}{2\sqrt{s-t}}.$$

We see that s varies from 0 to 36, and t varies from -1 to 0. The integrand is $x = \sqrt{s-t}$, so we get

$$\begin{aligned}
\iint_R x \, dA &= \int_0^{36} \int_{-1}^0 \sqrt{s-t} \frac{1}{2\sqrt{s-t}} \, dt \, ds \\
&= \int_0^{36} \int_{-1}^0 \frac{1}{2} \, dt \, ds \\
&= \int_0^{36} \frac{1}{2} \, ds = 18.
\end{aligned}$$

(c) Sketch the region of integration and compute $\iint_R xy^2 \, dx \, dy$, where R is the region in the first quadrant bounded by the curves $y = x^2$ and $x = y^2$. (Do not use a calculator!)

Solution: These curves intersect at $(0,0)$ and $(1,1)$. So we can take x going from 0 to 1, and then y goes from x^2 to $\sqrt{x}$. (Coutnerintuitively, $x^2 < \sqrt{x}$ in this region.)

So we compute

$$\begin{aligned}\iint_R xy^2\,dx &= \int_0^1 \int_{x^2}^{\sqrt{x}} xy^2\,dy\,dx \\ &= \int_0^1 xy^3/3\Big|_{x^2}^{\sqrt{x}}\,dx \\ &= \frac{1}{3}\int_0^1 x^{5/2} - x^7\,dx \\ &= \frac{1}{3}\left(\frac{2}{7}x^{7/2} - \frac{1}{8}x^8\right)\Big|_0^1 \\ &= \frac{1}{3}\left(2/7 - 1/8\right) = \frac{3}{56}.\end{aligned}$$

We could have set it up in the other order. Then we would get

$$\begin{aligned}\iint_R xy^2\,dx &= \int_0^1 \int_{y^2}^{\sqrt{y}} xy^2\,dx\,dy \\ &= \int_0^1 x^2y^2/2\Big|_{y^2}^{\sqrt{y}}\,dy \\ &= \frac{1}{2}\int_0^1 y^3 - y^6\,dy \\ &= \frac{1}{2}\left(y^4/4 - y^7/7\right)\Big|_0^1 \\ &= \frac{1}{2}(1/4 - 1/7) = \frac{3}{56}.\end{aligned}$$

I think the second way looks easier, but it's up to you! (When I worked this out, I got to the point where I was cubing $\sqrt{x}$ and realized if I did it in the other order things would work out better.)

S4: Integral Applications

(a) Find the mass of the tetrahedron bounded by the planes $x = 0, y = 0, z = 0$, and $x + 2y + 3z = 6$ if the density is given by $\delta(x, y, z) = z$.

Solution: z goes from 0 to 2. Then y goes from 0 to $6-3z/2$ and x goes from 0 to $6-2y-3z$, and we have

$$\begin{aligned}
\int_0^2\int_0^{3-3z/2}\int_0^{6-2y-3z} z\,dx\,dy\,dz &= \int_0^2\int_0^{3-3z/2} z\Big|_0^{6-2y-3z}\,dy\,dz \\
&= \int_0^2\int_0^{3-3z/2} 6z-2yz-3z^2\,dy\,dz \\
&= \int_0^2 6yz-y^2z-3yz^2\Big|_0^{3-3z/2}\,dz \\
&= \int_0^2 18z-9z^2-(3-3z/2)^2z-9z^2+9z^3/2\,dz \\
&= \int_0^2 18z-18z^2+9z^3/2-9z+9z^2-9z^3/4\,dz \\
&= \int_0^2 9z-9z^2+9z^3/4\,dz \\
&= 9z^2/2-3z^3+9z^4/16\Big|_0^2 = 18-24+9=3.
\end{aligned}$$

(b) Let R be a trapezoidal lamina bounded by the lines $y=-x/4+5/2, y=0, y=2, x=0$, with density $\rho(x,y)=y^2$.

(i) Sketch a picture of R.

(ii) Find the mass of R.

(iii) Find the center of mass of R.

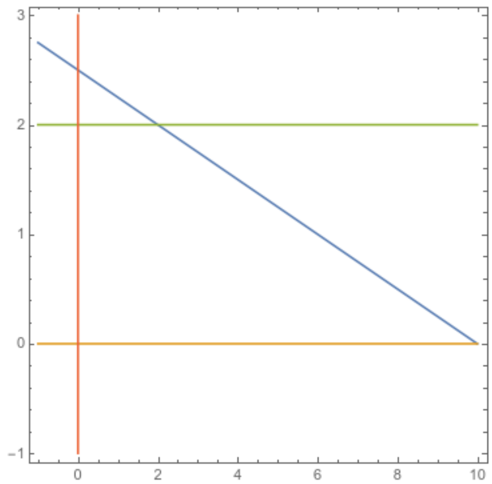
(iv) Find the moments of inertia of R.

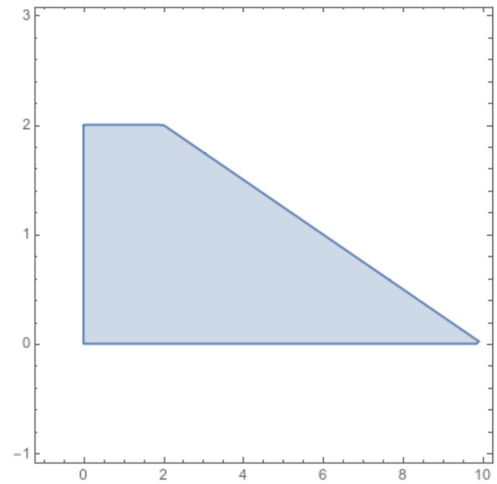
Solution:

(i)

(ii) We want to set up an integral over the region. So we see $0\le y\le 2$, and then $0\le x\le 10-4y$. So we integrate

$$\begin{aligned}
m &= \int_0^2\int_0^{10-4y} y^2\,dx\,dy \\
&= \int_0^2 10y^2-4y^3\,dy \\
&= \frac{10}{3}y^3-y^4\Big|_0^2 = \frac{80}{3}-16=\frac{32}{3}.
\end{aligned}$$



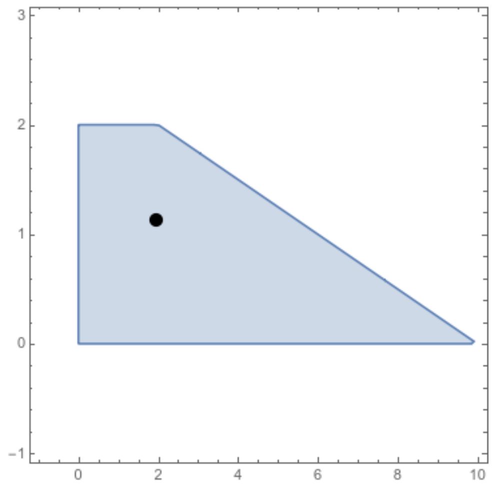


(iii) First we need to compute the first moments.

$$
\begin{aligned}
M_x &= \int_0^2 \int_0^{10-4y} y \cdot \rho(x,y)\, dx\, dy = \int_0^2 \int_0^{10-4y} y^3\, dx\, dy \\
&= \int_0^2 10y^3 - 4y^4\, dy = \frac{5}{2}y^4 - \frac{4}{5}y^5\Big|_0^2 \\
&= 40 - \frac{128}{5} = \frac{72}{5}. \\
M_y &= \int_0^2 \int_0^{10-4y} x \cdot \rho(x,y)\, dx\, dy = \int_0^2 \int_0^{10-4y} xy^2\, dx\, dy \\
&= \int_0^2 \frac{1}{2}x^2y^2\Big|_0^{10-4y} dy = \int_0^2 (50 - 40y + 8y^2)y^2\, dy \\
&= \int_0^2 50y^2 - 40y^3 + 8y^4 = \frac{50}{3}y^3 - 10y^4 + \frac{8}{5}y^5\Big|_0^2 \\
&= \frac{400}{3} - 160 + \frac{256}{5} = \frac{368}{15}.
\end{aligned}
$$

Thus the coordinates of the center of mass are

$$
\begin{aligned}
\overline{x} &= \frac{M_y}{m} = \frac{368/15}{32/3} = \frac{23}{10} \\
\overline{y} &= \frac{M_x}{m} = \frac{72/5}{32/3} = \frac{27}{20}
\end{aligned}
$$



(iv)

$$
\begin{aligned}
I_x &= \int_0^2 \int_0^{10-4y} y^2 \cdot \rho(x,y)\,dx\,dy = \int_0^2 \int_0^{10-4y} y^4\,dx\,dy \\
&= \int_0^2 10y^4 - 4y^5\,dy = 2y^5 - \frac{2}{3}y^6\Big|_0^2 \\
&= 64 - \frac{128}{3} = \frac{64}{3}. \\
I_x &= \int_0^2 \int_0^{10-4y} x^2 \cdot \rho(x,y)\,dx\,dy = \int_0^2 \int_0^{10-4y} x^2y^2\,dx\,dy \\
&= \int_0^2 \frac{1}{3}x^3y^2\Big|_0^{10-4y}\,dy \int_0^2 \frac{1}{3}(10-4y)^3y^2\,dy \\
&= \int_0^2 \frac{1000}{3}y^2 - 400y^3 + 160y^4 - \frac{64}{3}y^5\,dy \\
&= \frac{1000}{9}y^3 - 100y^4 + 32y^5 - \frac{32}{9}y^6\Big|_0^2 \\
&= \frac{8000}{9} - 1600 + 1024 - \frac{2048}{9} = \frac{256}{3}.
\end{aligned}
$$