Math 2233 Summer 2025
Multivariable Calculus
Mastery Quiz 9
Due Wednesday, July 30

This week's mastery quiz has three topics. Everyone should submit M5. If you have a 4/4 on M4, (meaning you have gotten 2/2 twice), or a 2/2 on S4 you don't need to do them again.

Don't worry if you make a minor error, but try to demonstrate your mastery of the underlying material. Feel free to consult your notes, but please **don't discuss the actual quiz questions with other students in the course**.

Remember that you are trying to demonstrate that you understand the concepts involved. For all these problems, justify your answers and show your work. Do not just write "yes" or "no" or give a single number.

Please turn this quiz in class on Wednesday. You may print this document out and write on it, or you may submit your work on separate paper; in either case make sure your name and recitation section are clearly on it. If you absolutely cannot turn it in in person, you can submit it electronically but this should be a last resort.

Topics on This Quiz

- Major Topic 4: Integration
- Major Topic 5: Line Integrals
- Secondary Topic 4: Integral Applications

Name:

Name:

M4: Integration

(a) Find the volume of the solid bounded by the paraboloid $z = x^2 + y^2$ and the sphere $x^2 + y^2 + z^2 = 6$ (above the plane z=0).

Solution: We see that the two surfaces intersect at $z+z^2 = 6$, which has the solutions $z = 2$ and $z = -3$. But we can't have $z = -3$ when $z = x^2 + y^2$, so the intersection is the circle at $z = 2$ where $x^2 + y^2 = 2$.

Converting to cylindrical coordinates, this gives us $r^2 = 2$. So we have r ranging from 0 to $\sqrt{2}$, with θ ranging from 0 to 2π, and then z ranges from x^2+y^2 to $\sqrt{6-x^2-y^2}$, which is really r^2 to $\sqrt{6-r^2}$. This gives

$$\begin{aligned}
\iiint_R 1\,dV &= \int_0^{2\pi}\int_0^{\sqrt{2}}\int_{r^2}^{\sqrt{6-r^2}} r\,dz\,dr\,d\theta \\
&= \int_0^{2\pi}\int_0^{\sqrt{2}} r\sqrt{6-r^2} - r^3\,dr\,d\theta \\
&= \int_0^{2\pi} \frac{-1}{3}(6-r^2)^{3/2} - \frac{r^4}{4}\Big|_0^{\sqrt{2}}\,d\theta \\
&= \int_0^{2\pi} \frac{-1}{3}4^{3/2} - 1 - \frac{-1}{3}6^{3/2}\,d\theta \\
&= \int_0^{2\pi} -11/3 + \sqrt{24}\,d\theta \\
&= (\sqrt{24} - 11/3)2\pi.
\end{aligned}$$

(b) Use a change of variables to integrate the function $(x+y)(x-y)$ over the diamond bounded by $x+y = 2, x+y = 0, x-y = 0, x-y = 2$.

Solution: We want to reparametrize this with $s = x+y, t = x-y$. Our bounds are $t = 2, s = 0, t = 2, t = 0$, and our function is st.

To compute the Jacobian we need to find x and y as a function of s and t. We see that $s+t = 2x$ so $x = \frac{s+t}{2}$; similarly, $s-t = 2y$ so $y = \frac{s-t}{2}$. Then the Jacobian is

$$\frac{\partial(x,y)}{\partial(s,t)} = \begin{vmatrix} \frac{\partial x}{\partial s} & \frac{\partial x}{\partial t} \\ \frac{\partial y}{\partial x} & \frac{\partial y}{\partial t} \end{vmatrix} = \begin{vmatrix} 1/2 & 1/2 \\ 1/2 & -1/2 \end{vmatrix} = -1/4 - 1/4 = -1/2$$

so $\left|\frac{\partial(x,y)}{\partial(s,t)}\right| = 1/2$. Then the integral is

$$\begin{aligned}
\int_R (x+y)(x-y)\,dA &= \int_0^2\int_0^2 st/2\,ds\,dt \\
&= \int_0^2 s^2t/4|_0^2\,dt = \int_0^2 t\,dt \\
&= t^2/2|_0^2 = 2.
\end{aligned}$$

Name:

M5: Line Integrals

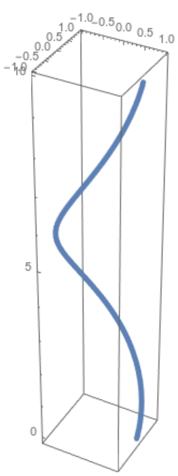
(a) Compute the line integral $\int_C f(x,y)\,ds$ for $f(x,y) = xy$ where C is the first quarter of the unit circle (starting at $(1,0)$ and ending at $(0,1)$).

Solution: We have $\vec{r}(t) = (\cos(t), \sin(t))$ for $t \in [0, \pi/2]$, so

$$\begin{aligned}
\int_C f\,ds &= \int_0^{\pi/2} f(\vec{r}(t)) \cdot \|\vec{r}'(t)\|\,dt \\
&= \int_0^{\pi/2} \cos(t)\sin(t)\|\langle -\sin(t), \cos(t)\rangle\|\,dt \\
&= \int_0^{\pi/2} \cos(t)\sin(t)\sqrt{\sin^2(t)+\cos^2(t)}\,dt \\
&= \int_0^{\pi/2} \cos(t)\sin(t)\,dt \\
&= \frac{1}{2}\sin^2(t)\Big|_0^{\pi/2} = \frac{1}{2}(\sin^2(\pi/2) - \sin^2(0)) = \frac{1}{2}\cdot 1^2 = 1/2.
\end{aligned}$$

(b) Let C be the helix that winds around the cylinder $x^2+y^2 = 1$ (counterclockwise viewed from the positive z-axis looking down on the xy-plane), starting at $(1,0,0)$, winding around the cylinder once, and ending at the point $(1,0,1)$.

Compute the line integral of the vector field $\vec{F}(x,y,z) = (-y, x, z^2)$.



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Solution: We parametrize the curve with $\vec{r}(t) = (\cos(t), \sin(t), \frac{t}{2\pi})$. Then we have

$$\begin{aligned}
\int \vec{F} \cdot d\vec{r} &= \int_0^{2\pi} \left(-\sin(t), \cos(t), \frac{t^2}{4\pi^2}\right) \cdot \left(-\sin(t), \cos(t), \frac{1}{2\pi}\right) dt \\
&= \int_0^{2\pi} \sin^2(t) + \cos^2(t) + \frac{t^2}{8\pi^3} dt \\
&= \int_0^{2\pi} 1 + \frac{t^2}{8\pi^3} dt \\
&= t + \frac{t^3}{24\pi^3}\Big|_0^{2\pi} = 2\pi + \frac{8\pi^3}{24\pi^3} = 2\pi + \frac{1}{3}.
\end{aligned}$$

S4: Integral Applications

(a) Suppose an insurer has found that if x is the age of a car involved in an accident, and y is the length of time since the car was insured, then these variables have a joint probability density function of

$$p(x,y) = \begin{cases} c(5-xy) & 1 \le x \le 5 \quad \text{and} \quad 0 \le y \le 1 \\ 0 & \text{otherwise} \end{cases}$$

(i) Find the value of c that makes this a probability density function.

Solution:

$$\begin{aligned}
P &= \int_0^1 \int_1^5 c(5-xy)\,dx\,dy = c\int_0^1 5x - \frac{x^2y}{2}\Big|_1^5 dy \\
&= c\int_0^1 25 - \frac{25}{2}y - 5 + \frac{1}{2}y\,dy = c\int_0^1 20 - 12y\,dy \\
&= c\left(20y - 6y^2\right)\Big|_0^1 = c(20-6) = c(14) = 1.
\end{aligned}$$

Thus we have $c = \frac{1}{14}$.

(ii) What is the expected age of a car involved in an accident?

Solution: The age of the car is x so we're looking for the expected value of x. Thus we compute

$$\begin{aligned}
\overline{x} &== \frac{1}{14}\int_0^1 \int_1^5 x(5-xy)\,dx\,dy = \frac{1}{14}\int_0^1 \int_1^5 5x - x^2y\,dx\,dy \\
&= \frac{1}{14}\int_0^1 \frac{5}{2}x^2 - \frac{1}{3}x^3y\Big|_1^5 dy \\
&= \frac{1}{14}\int_0^1 \frac{125}{2} - \frac{125}{3}y - \frac{5}{2} + \frac{1}{3}y\,dy = \frac{1}{14}\int_0^1 60 - \frac{124}{3}y\,dy \\
&= \frac{1}{14}\left(60y - \frac{62}{3}y^2\right)\Bigg|_0^1 = \frac{1}{14}\left(60 - \frac{62}{3}\right) = \frac{118}{42} = \frac{59}{21}.
\end{aligned}$$

(iii) What is the average duration of the insurance policy at the time of the accident?

Solution: The duration of the policy is y so we're looking for the expected value of y. Thus we compute

$$\begin{aligned}
\overline{y} &== \frac{1}{14}\int_0^1\int_1^5 y(5-xy)\,dx\,dy = \frac{1}{14}\int_0^1\int_1^5 5y - xy^2\,dx\,dy \\
&= \frac{1}{14}\int_0^1 5xy - \frac{1}{2}x^2y^2\Big|_1^5\,dy \\
&= \frac{1}{14}\int_0^1 25y - \frac{25}{2}y^2 - 5y + \frac{1}{2}y^2\,dy = \frac{1}{14}\int_0^1 20y - 12y^2\,dy \\
&= \frac{1}{14}\left(10y^2 - 4y^3\right)\Big|_0^1 = \frac{1}{14}(10-4) = \frac{6}{14} = \frac{3}{7}.
\end{aligned}$$

(b) Set up (but do not evaluate) an integral to compute the mass of the region inside the cylinder $y^2+z^2=1$ between the planes $x+y=3$ and $x-z=7$, if the density is $\delta(xyz)=e^{-x-yz}$.

Solution: We can take y from -1 to 1, and then z from $-\sqrt{1-y^2}$ to $\sqrt{1-y^2}$. So we just need to figure out the bounds on x. But we have $x=7+z$ on one side and $x=3-y$ on the other, so we get the integral

$$\int_{-1}^1\int_{-\sqrt{1-y^2}}^{\sqrt{1-y^2}}\int_{3-y}^{7+z} e^{-x-yz}\,dx\,dz\,dy.$$