# Jefferson's Method

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# Jefferson's Method

**Definition (Jefferson's method)**

- Choose a modified divisor $d$
- Compute the modified quotas $p_k/d$
- Round these down to obtain $a_k = \lfloor p_k/d \rfloor$.
- If $a_1 + a_2 + \cdots + a_n = h$, then we have the Jefferson apportionment.
- Otherwise, choose a new $d$ and try again.

# Jefferson's Method

**Example**

Find the Jefferson apportionment with $n = 3, h = 10$, and state populations $p_1 = 1{,}500$, $p_2 = 3{,}200$, $p_3 = 5{,}300$.

| | | $s$ =1,000 | | $d = 900$ | | $d = 800$ | | $d = 850$ | |
|---|---|---|---|---|---|---|---|---|---|
| $k$ | $p_k$ | $q_k$ | $\lfloor q_k \rfloor$ | $q$ | $\lfloor q \rfloor$ | $q$ | $\lfloor q \rfloor$ | $q$ | $\lfloor q \rfloor$ |
| 1 | 1,500 | 1.50 | 1 | $1.\overline{66}$ | 1 | 1.875 | 1 | 1.76 | 1 |
| 2 | 3,200 | 3.20 | 3 | $3.\overline{55}$ | 3 | 4 | 4 | 3.76 | 3 |
| 3 | 5,300 | 5.30 | 5 | $5.\overline{88}$ | 5 | 6.625 | 6 | 6.24 | 6 |
| | 10,000 | | 9 | | 9 | | 11 | | 10 |

# Questions about Jefferson's Method

1. Is there always a $d$ that will work?
   - Yes, if we don't have ties
2. Can we find it?
   - Yes!
   - Guess a random $d$
   - If it gives too many seats, pick a larger number
   - If it doesn't give enough seats, pick a smaller number
3. Is there ever *more than one* $d$ that will work?
   - Yes
4. If we pick two different $d$s that both give the same total number of seats, will they give the same apportionment?
   - Fortunately, yes.

# Jefferson's Method

## Proposition

*Suppose $h, n$, and $p_1, \ldots, p_n$ are given as inputs to our apportionment function. If $d$ and $d'$ are two different divisors, yielding Jefferson apportionments $a_1, \ldots, a_n$ and $a'_1, \ldots, a'_n$ respectively, then $a_k = a'_k$ for each state $k$.*

## Proof.

- Suppose "without loss of generality" that $d \leq d'$.
- For every state, $p_k/d \geq p_k/d'$.
- Rounding down doesn't change that, so $a_k \geq a'_k$.
- Both apportionments give $h$ seats, so

  $a_1 + a_2 + \cdots + a_n = a'_1 + a'_2 + \cdots + a'_n$.

# Jefferson's Method

## Proposition

*Suppose $h, n$, and $p_1, \ldots, p_n$ are given as inputs to our apportionment function. If $d$ and $d'$ are two different divisors, yielding Jefferson apportionments $a_1, \ldots, a_n$ and $a'_1, \ldots, a'_n$ respectively, then $a_k = a'_k$ for each state $k$.*

## Proof.

- For every state, $a_k \geq a'_k$
- $h = a_1 + a_2 + \cdots + a_n \geq a'_1 + a'_2 + \cdots + a'_n = h$
- No number on the left can be smaller, so none of them can be bigger.
- Therefore, for each $k$, $a_k = a'_k$.

# How do we find the right divisor?

## Trial and error

- Guess randomly
- Will work eventually probably.

## Binary search

- Guess a divisor
- If it's too big, pick a smaller one
- If it's too small, pick a bigger one
- If one is too small and another is too big, try halfway between
- Keep trying halfway points until one works.

- There's a better way
- Understand *why* a divisor will work

# Critical Divisors

- Small changes in $d$ rarely matter

| | | $d = 850$ | | $d = 860$ | | $d = 870$ | | $d = 880$ | | $d = 890$ | |
|---|---|---|---|---|---|---|---|---|---|---|---|
| $k$ | $p_k$ | $q$ | $\lfloor q \rfloor$ | $q$ | $\lfloor q \rfloor$ | $q$ | $\lfloor q \rfloor$ | $q$ | $\lfloor q \rfloor$ | $q$ | $\lfloor q \rfloor$ |
| 1 | 1,500 | 1.76 | 1 | 1.74 | 1 | 1.72 | 1 | 1.70 | 1 | 1.69 | 1 |
| 2 | 3,200 | 3.67 | 3 | 3.72 | 3 | 3.68 | 3 | 3.64 | 4 | 3.60 | 4 |
| 3 | 5,300 | 6.24 | 6 | 6.16 | 6 | 6.09 | 6 | 6.02 | 6 | 5.96 | 5 |
| | 10,000 | | 10 | | 10 | | 10 | | 10 | | 9 |

- When will it change?

# Critical Divisors

- Had $p_3 = 5{,}300$
- If $d = 850$ then $q = \frac{5{,}300}{850} \approx 6.24$.
- 6.24 "rounds down to" 6. What does that mean?
- $6 \leq 6.24 < 7$ that is, $6 \leq \frac{5{,}300}{850} < 7$
- $850 \leq \frac{5{,}300}{6}$ but also $\frac{5{,}300}{7} < 850$.
- When does the rounding change?
  - When $d = \frac{5{,}300}{6} = 883.3$ or $d = \frac{5{,}300}{7} = 757$.

**Definition**

We call a number of the form $\frac{p_k}{m}$ for a positive integer $k$ a (Jefferson) critical divisor for the state $k$.

# Critical Divisors

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We call a number of the form $\frac{p_k}{m}$ for a positive integer $k$ a (Jefferson) critical divisor for the state $k$.

- Imagine $d$ is very very large. What happens?
  - Every state gets no representatives
- Decrease $d$ slowly. When will a state first get a representative?
  - When $d$ gets less than the population.
  - That means the state contains a whole district.
- When will a state get its second representative?
  - When the state contains two districts
  - When $d$ is less than *half* the population.
- When will that state get its third representative?

# Critical Divisors

## Definition

We call a number of the form $\frac{p_k}{m}$ for a positive integer $k$ a (Jefferson) critical divisor for the state $k$.

- Suppose $p_3 = 5,300$.

| $m$ | 1 | 2 | 3 | 4 | 5 | 6 |
|---|---|---|---|---|---|---|
| $\frac{p_k}{m}$ | 5,300 | 2,650 | 1,767 | 1,325 | 1,060 | 883 |

- If $d > 5,300$, then $a_k = 0$.
- If $5,300 > d > 2,650$, then $a_k = 1$
- If $2,650 > d > 1,767$, then $a_k = 2$
- When will $a_k = 5$?
  - When $1,060 > d > 883$.

# Critical Divisors

## Computing with Critical Divisors

- If $d$ is very big, give zero seats
- Shrink $d$. Every time we cross a critical divisor for a state, that state gets one more representative.
- We want to give out $h$ seats in total
- So we give seats to the biggest $h$ critical divisors
- Can pick any $d$ between divisors $h$ and $h+1$.

# Critical Divisors

## Example

Find the Jefferson apportionment with $n = 3$, $h = 10$, and state populations $p_1 = 1{,}500$, $p_2 = 3{,}200$, $p_3 = 5{,}300$.

| d | State 1 | State 2 | State 3 |
|---|---|---|---|
| 1 | $\frac{1{,}500}{1} = 1,500$ | $\frac{3{,}200}{1} = 3,200$ | $\frac{5{,}300}{1} = 5,300$ |
| 2 | $\frac{1{,}500}{2} = 750$ | $\frac{3{,}200}{2} = 1,600$ | $\frac{5{,}300}{2} = 2,650$ |
| 3 | $\frac{1{,}500}{3} = 500$ | $\frac{3{,}200}{3} = 1,067$ | $\frac{5{,}300}{3} = 1,767$ |
| 4 | $\frac{1{,}500}{4} = 375$ | $\frac{3{,}200}{4} = 800$ | $\frac{5{,}300}{4} = 1,325$ |
| 5 | $\frac{1{,}500}{5} = 300$ | $\frac{3{,}200}{5} = 640$ | $\frac{5{,}300}{5} = 1,060$ |
| 6 | $\frac{1{,}500}{6} = 250$ | $\frac{3{,}200}{6} = 533$ | $\frac{5{,}300}{6} = 883$ |
| 7 | $\frac{1{,}500}{7} = 215$ | $\frac{3{,}200}{7} = 457$ | $\frac{5{,}300}{7} = 757$ |

# Critical Divisors

## Advantages and Disadvantages

- No trial and error
- Straightforward calculation
- A *lot* of straightforward calculation

## Can we speed this up?

- Can calculate $s$ easily
- Good first guess, but $d$ should always be smaller
- Count down from $s$.

# Critical Divisors

## Example

If $n = 3, h = 10$, $p_1 = 1{,}500$, $p_2 = 3{,}200$, and $p_3 = 5{,}300$ then

$p = 10{,}000$ and $s = 1{,}000$

| d | State 1 | State 2 | State 3 |
|---|---|---|---|
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