Evaluating Voting Methods I

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Summary

Voting Methods

- Plurality
- dictatorship
- Copeland's Method
- Hare's method
- monarchy
- Borda count
- Coombs's Method
- all ties
- More to come!

Voting Method Criteria

- unanimous
- anonymous
- Pareto
- Condorcet
- decisive
- neutral
- independent
- anti-Condorcet
- majoritarian
- monotone

Independence

Definition

- A method is independent if it satisfies the following (somewhat complicated) property:
- Suppose there are two profiles where no voter changes their mind about whether candidate A is preferred to candidate B: if a voter ranks A above B in the first profile, they also rank A above B in the second profile
- If A a wins in the first profile but B doesn't, then B cannot win in the second profile.
- (This also works backwards: if B wins in the second profile, they can't lose in the first.)

Independence

- Kenneth Arrow (1950)
- "Independence of irrelevant alternatives" or "IIA"

After finishing dinner, [Columbia philosopher] Sidney Morgenbesser decides to order dessert. The waitress tells him he has two choices: apple pie and blueberry pie. Sidney orders the apple pie. After a few minutes the waitress returns and says that they also have cherry pie at which point Morgenbesser says "In that case I'll have the blueberry pie."

Independence

Example

Profile 1						
A	C	B	C	C	A	B
B	A	C	B	B	B	A
C	B	A	A	A	C	C

Profile 2						
A	C	B	C	C	A	B
B	A	C	B	B	B	C
C	B	A	A	A	C	A

- Relative ranks of A and B haven't changed.
- If A wins and B loses in Profile 1, then B shouldn't win in Profile 2.

Independence: Florida 2000

Example (Florida Presidential Vote in 2000)

2,912,790	2,912,253	97,488
Bush (R)	Gore (D)	Nader (G)

Guessed ranked choice		
2,912,790	2,912,253	97,488
Bush	Gore	Nader
Gore	B/N	Gore
Nader	N/B	Bush

What if?		
2,912,790	2,912,253	97,488
Bush	Gore	Gore
Gore	B/N	Nader
Nader	N/B	Bush

- Bush won by plurality
- Changing Gore/Nader preferences could cause Bush to lose

Independence

- Why is the phrasing so complicated?
- *Idea*: if A beats B in profile 1, A still beats B in profile 2.
- But a move could make A and B both lose.

Example

Profile 1		
A	A	C
C	C	A
B	B	B

Profile 2		
A	C	C
C	A	A
B	B	B

- Change shouldn't make B win but it can make A lose
- We don't name second place, so A doesn't "beat" B.

Our First Impossibility Result

Proposition

Any social choice function that satisfies anonymity and neutrality must violate decisiveness.

Proof.

- Since our method is anonymous, can consider tabulated profiles
- Suppose we have $2n$ voters and 2 candidates
- Consider two profiles:

n	n
A	B
B	A

n	n
B	A
A	B

Our First Impossibility Result

Proposition

Any social choice function that satisfies anonymity and neutrality must violate decisiveness.

Proof.

n	n
A	B
B	A

n	n
B	A
A	B

- At least one candidate wins.
- Assume A wins in the top profile.
- By neutrality B wins in the bottom.
- Profiles are the same, so B wins in top.
- Top profile does not have a unique winner.
- Therefore the method can't be decisive.

Taylor's Theorem

Proposition (Taylor)

No social choice function involving at least three candidates satisfies both independence and the Condorcet criterion.

Proof.

- Suppose we have an independent Condorcet method.
- Consider this profile:

A	C	B
B	A	C
C	B	A

- Claim no candidate can be a winner.

□

Taylor's Theorem

A	C	B
B	A	C
C	B	A

A	C	C
B	A	B
C	B	A

Claim

A can't win under independence and Condorcet

Proof.

- Consider this profile 2
- By Condorcet, C must be unique winner in profile 2
- C wins and A loses in profile 2
- Only swapped C and B, so by independence A loses in profile 1.

□

Taylor's Theorem

Proposition (Taylor)

No social choice function involving at least three candidates satisfies both independence and the Condorcet criterion.

Proof.

A	C	B
B	A	C
C	B	A

- Assume we have an independent Concorcet method for this profile
- Showed A can't win this profile
- Same argument shows B can't win
- Same argument shows C can't win
- Our method can't name a winner for this profile.

Proposition

If a method is Condorcet then it satisfies the majority criterion.

Proof.

- Suppose A has a majority of first-place votes.
- Then A will win any head-to-head matchup, so A is the Condorcet candidate.
- Any method that satisfies the Condorcet criterion will cause A to win, also satisfying the majority criterion.

□

Remark

We can say the Condorcet criterion is "stronger" than the majority criterion.

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Analyzing the Plurality Method

Proposition

The plurality method is majoritarian, monotone, and Pareto, but not Condorcet, anti-Condorcet, or independent.

Proof.

- The majority is always a plurality, so candidate with majority will win.
- Monotone: raising a candidate on some preference lists can't reduce their first-place votes, or increase anyone else's
- Pareto: if A is ahead of B on every preference list, then B gets no votes, and can't win.

Analyzing the Plurality Method

Claim

The plurality method is neither Condorcet not anti-Condorcet.

Proof.

2	3	2
A	B	C
C	A	A
B	C	B

- B wins plurality, but
- A beats B, A beats C, and C beats B.
- A is Condorcet, and loses
- B is anti-Condorcet, and wins

□

Analyzing the Plurality Method

Claim

The plurality method is not independent.

Proof.

A	A	A	A	B	B	B
B	B	C	C	A	A	A
C	C	B	B	C	C	C

$\rightarrow$

A	A	C	C	B	B	B
B	B	A	A	A	A	A
C	C	B	B	C	C	C

- First profile: A wins, B and C lose
- Second profile: B wins, A and C lose
- But relative position of A and B doesn't change.