Evaluating Voting Methods III

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Summary

Voting Methods

- Plurality
- dictatorship
- Copeland's Method
- Hare's method
- monarchy
- Borda count
- Coombs's Method
- all ties
- Antiplurality

Voting Method Criteria

- unanimous
- anonymous
- Pareto
- Condorcet
- decisive
- neutral
- independent
- anti-Condorcet
- majoritarian
- monotone

Previous Results

Proposition

The plurality method is majoritarian, monotone, and Pareto, but not Condorcet, anti-Condorcet, or independent.

Proposition

The antiplurality method is monotone, but not majoritarian, Condorcet, anti-Condorcet, Pareto, or independent.

Proposition

Hare's method is majoritarian and Pareto, but not monotone, Condorcet, anti-Condorcet, or independent.

Proposition

Coombs's method is Pareto, but not majoritarian, monotone, Condorcet, anti-Condorcet, or independent.

Borda Count

Definition

If there are n candidates, give $n-1$ points for a first-place vote, $n-2$ for a second-place vote, down to 0 for a last-place vote. The candidate(s) with the most votes win.

Discussion Question

What criteria will the Borda Count satisfy?

Borda Count

Claim

The Borda count is monotone.

Proof.

- Raising a candidate on preference lists can't reduce their score
- Raising a candidate on preference lists can't raise anyone else's score
- A winner will still win after rising on some lists.

□

Borda Count

Claim

The Borda count is Pareto.

Proof.

- If every voter prefers A to B, each voter will give A more points than B.
- A will get a higher score than B, so B can't win.

□

Borda Count

Claim

The Borda count is not Condorcet or majoritarian.

Proof.

Consider:

A	A	A	B	B
B	B	B	C	C
C	C	C	A	A

- What happens?
  - A gets 6 points
  - B gets 7 points
  - C gets 2 points
  - B wins
- A is the Condorcet candidate.
- A is the majority candidate.

□

Borda Count

Claim

The Borda count is not independent.

Proof.

C	A	A	B	B
A	B	B	A	A
B	C	C	C	C

$\rightarrow$

C	A	A	B	B
A	B	B	C	C
B	C	C	A	A

- Profile 1: A gets 7, B gets 6, C gets 2. A wins
- Profile 2: A gets 5, B gets 6, C gets 4. B wins
- Only changed relative positions of A and C.

Borda Count

Claim

The Borda count is anti-Condorcet.

- First time we've proven something *is* anti-Condorcet
- New type of argument
- Can't just give an example
- Kind of complicated!

Borda Count

Claim

The Borda count is anti-Condorcet.

Proof.

- Suppose n candidates and m voters
- Each voter gives $\frac{n(n-1)}{2}$ total votes
- Total number of points: $m \cdot \frac{n(n-1)}{2}$.
- Average number of points per candidate is

$$\frac{mn(n-1)}{2n} = \frac{m(n-1)}{2}.$$

Borda Count

Claim

The Borda count is anti-Condorcet.

Proof.

- Average score: $\frac{m(n-1)}{2} = \frac{1}{2}m(n-1)$
- Some candidate will be at least average
- Max score: $m(n-1)$
- Average score is half of max score
- Want to show an anti-Condorcet candidate gets less than that.

Borda Count

Claim

Let A be an anti-Condorcet candidate. Then A's Borda count will be less than half the maximum possible.

Proof.

- New perspective: get one Borda point each time one voter ranks you ahead of one other candidate.
- A gets ranked below each other candidate more than half the time
- A gets less than half the possible points
- A gets less than $\frac{1}{2}m(n-1)$ total points.

Borda Count

Claim

The Borda count is anti-Condorcet.

Proof.

- Let A be an anti-Condorcet candidate.
- Then A gets less than $\frac{1}{2}m(n-1)$ total points.
- But the average score is $\frac{1}{2}m(n-1)$ points.
- At least one candidate will do average or better, so some candidate gets more points than A.
- If A is anti-Condorcet then A cannot win in the Borda count.

Borda Count

Discussion Question

Why doesn't the same argument show the Borda count is Condorcet?

- We showed an anti-Condorcet candidate can't come in *first*
- Same argument shows a Condorcet candidate can't come in *last*
- But the anti-Condorcet candidate isn't always last, and a Condorcet candidate isn't always first.

Proposition

The Borda count method is monotone, anti-Condorcet, and Pareto, but not majoritarian, Condorcet, or independent.

Copeland's method

Definition

Each candidate earns one point for every candidate they beat in a head-to-head matchup (using a simple majority method). A candidate earns half a point for every candidate they tie. The candidate(s) with the most points at the end win.

Discussion Question

What criteria will Copeland's method satisfy?

Copeland's method

Claim

Copeland's method is Condorcet, majoritarian, and anti-Condorcet.

Proof.

- A Condorcet candidate wins each head-to-head and gets a perfect score
- No one else can get a perfect score, so the Condorcet candidate is the unique winner
- This means a majority candidate is also the unique winner
- An anti-Condorcet candidate loses each matchup
- Gets zero points and can't win

□

Copeland's method

Claim

Copeland's method is Pareto.

Proof.

- Suppose A is above B on every preference list
- Then A wins every matchup B wins
- A gets a point whenever B gets a point
- A gets at least half a point whenever B gets half a point
- A beats B, so gets a point B doesn't get
- A scores more than B, so B can't win.

□

Copeland's method

Claim

Copeland's method is monotone.

Proof.

- Moving A up on some lists won't hurt them in any head-to-head, so won't reduce A's score
- Won't affect any other head-to-head at all
- So it can't increase any other candidate's score
- If A wins before the switch, will also win after.

□

Discussion Question

How does this suggest we look at independence?

Copeland's method

Claim

Copeland's method is monotone.

Proof.

- Moving A up on some lists won't hurt them in any head-to-head, so won't reduce A's score
- Won't affect any other head-to-head at all
- So it can't increase any other candidate's score
- If A wins before the switch, will also win after.

□

Discussion Question

How does this suggest we look at independence?

Copeland's method

Claim

Copeland's method is not independent.

Proof.

B	A	A
C	B	C
A	C	B

$\rightarrow$

B	A	C
C	B	A
A	C	B

- Profile 1: A gets 2, B gets 1, C gets 0. A wins.
- Profile 2: A gets 1, B gets 1, C gets 1. All candidates win
- A and B haven't changed relative positions
- B loses in profile 1 and wins in profile 2

Copeland's Method

Proposition

Copeland's method is majoritarian, Condorcet, anti-Condorcet, monotone, and Pareto, but not independent.

Black's method

- Duncan Black (1908-1991)
- *The Theory of Committees and Elections* (1958)

Definition

Black's method is the social choice function that chooses the Condorcet candidate as the unique winner if there is a Condorcet candidate, and chooses the Borda count winner if there is not.

- Attempt to combine benefits of multiple methods
- Start with Borda count, "fix" the "problem" that it's not Condorcet.

Discussion Question

What criteria will Black's method satisfy?

Black's method

Claim

Black's method is Condorcet and majoritarian.

Proof.

- Condorcet by definition.
- Majoritarian because Condorcet.

□

Claim

Black's method is anti-Condorcet

Proof.

- anti-Condorcet candidate isn't Condorcet
- anti-Condorcet candidate can't win Borda count

□

Black's Method

Claim

Black's method is monotone.

Proof.

- Two cases
- If A is Condorcet winner, moving them up in some rankings won't change that, so they still win.
- If no Condorcet winner and A wins by Borda count:
  - Moving A up can't lower their score or raise anyone else's score, so they win Borda count
  - Moving A up can't make anyone *else* into Condorcet winner
  - Moving A up could make A the Condorcet winner, but that's fine.

□

Black's Method

Claim

Black's method is Pareto.

Proof.

- Suppose all voters prefer A to B
- B is not the Condorcet candidate since they lose to A
- B can't win the Borda count because A will have more points
- B can't win.

□

Black's Method

Claim

Black's method is not independent.

Proof.

- Easy answer: it's Condorcet, so not independent. Or:

B	A	A		B	A	C
C	B	C	$\rightarrow$	C	B	A
A	C	B		A	C	B

- Profile 1: A is the Condorcet candidate, and wins
- Profile 2: no Condorcet candidate
  - Each candidate gets 3 Borda points
  - All three are winners

Black's Method

Proposition

Black's method is majoritarian, Condorcet, anti-Condorcet, monotone, and Pareto, but not independent.

Tradeoffs

Poll Question

- Which properties have been common?
- Which have been uncommon?

Discussion Question

- Why is independence so uncommon?
- Do any methods achieve independence?