The Condorcet Paradox and Arrow's Theorem

Jay Daigle

jaydaigle@gwu.edu

https://jaydaigle.net/politics

The George Washington University

Previous Results

Proposition

The plurality method is majoritarian, monotone, and Pareto, but not Condorcet, anti-Condorcet, or independent.

Proposition

The antiplurality method is monotone, but not majoritarian, Condorcet, anti-Condorcet, Pareto, or independent.

Proposition

Hare's method is majoritarian and Pareto, but not monotone, Condorcet, anti-Condorcet, or independent.

Proposition

Coombs's method is Pareto, but not majoritarian, monotone, Condorcet, anti-Condorcet, or independent.

Previous Results

Proposition

The Borda count method is monotone, anti-Condorcet, and Pareto, but not majoritarian, Condorcet, or independent.

Proposition

Copeland's method is majoritarian, Condorcet, anti-Condorcet, monotone, and Pareto, but not independent.

Proposition

Black's method is majoritarian, Condorcet, anti-Condorcet, monotone, and Pareto, but not independent.

Tradeoffs

Poll Question

- Which properties have been common?
- Which have been uncommon?

Discussion Question

- Why is independence so uncommon?
- Do any methods achieve independence?

Dictatorship

Definition

One voter is the dictator. Their first choice is the unique winner.

Discussion Question

What criteria does the dictatorship method satisfy?

Dictatorship

Claim

The dictatorship method is monotone.

Proof.

- If A wins, they're at the top of the dictator's preference list
- Moving them up on other lists won't change that.

□

Dictatorship

Claim

The dictatorship method is Pareto.

Proof.

- If A is higher than B on every list, then A is higher on the dictator's list
- B isn't at the top of the dictator's list, and can't win.

□

Dictatorship

Claim

The dictatorship method is independent.

Proof.

- Suppose A wins and B loses in profile 1.
- Then A is at the top of the dictator's preference list in profile 1.
- If profile 2 has A and B in the same relative positions, then B is not at the top of the dictator's preference list in profile 2.
- B can't win in profile 2.

□

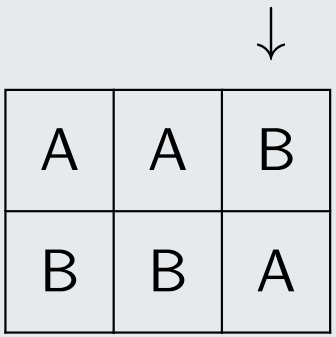
Dictatorship

Claim

The dictatorship method is not Condorcet, anti-Condorcet, or majoritarian.

Proof.

Consider:



- What happens?
- B wins
- A is the Condorcet candidate
- A is the majority candidate
- B is the anti-Condorcet candidate.

□

Dictatorship

Proposition

The dictatorship method is monotone, Pareto, and independent, but not Condorcet, anti-Condorcet, or majoritarian.

Constant Functions

Proposition

The all-ties method and the monarchy method are monotone and independent, but not Condorcet, anti-Condorcet, majority, or Pareto.

Proof.

- Constant functions: same output for any input
- Monotone and independent, because no candidate can win in one profile but lose in another
- Violate Condorcet, anti-Condorcet, majority, and Pareto, because rankings have no effect on who wins.

□

Summary

	anon	neu	unan	dec	maj	Con	AC	mono	Par	ind
Plurality	Y	Y	Y	N	Y	N	N	Y	Y	N
Antiplur	Y	Y	Y	N	N	N	N	Y	N	N
Borda	Y	Y	Y	N	N	N	Y	Y	Y	N
Hare	Y	Y	Y	N	Y	N	N	N	Y	N
Coombs	Y	Y	Y	N	N	N	N	N	Y	N
Copeland	Y	Y	Y	N	Y	Y	Y	Y	Y	N
Black	Y	Y	Y	N	Y	Y	Y	Y	Y	N
Dictator	N	Y	Y	Y	N	N	N	Y	Y	Y
All-ties	Y	Y	N	N	N	N	N	Y	N	Y
Monarchy	Y	N	N	Y	N	N	N	Y	N	Y

Tradeoffs

Proposition (Taylor)

No social choice function involving at least three candidates satisfies both independence and the Condorcet criterion.

- Condorcet isn't *that* common

Discussion Question

Why is independence so hard?

The Condorcet Paradox

Example

A	C	B
B	A	C
C	B	A

- Who should win?

Definition

- A Condorcet Paradox occurs when every candidate loses to at least one other candidate in a simple-majority head-to-head matchup.
- In this case we will get a cycle of candidates such that each beats the next head-to-head, until the last beats the first.

The Condorcet Paradox

Example

146	145	144
D	C	B
E	D	C
F	E	D
G	F	E
A	G	F
B	A	G
C	B	A

- Suppose we start with policy D
- Propose switch to C
- Passes with 289 votes
- Switch to B passes with 290 votes
- Switch to A passes with 291 votes
- Everyone is less happy!

Tradeoffs

Proposition (Taylor)

No social choice function involving at least three candidates satisfies both independence and the Condorcet criterion.

Discussion Question

Condorcet isn't that common. Why is independence so hard?

Theorem (Arrow's Impossibility Theorem)

If a social choice function with at least three candidates satisfies both Pareto and independence, then it must be a dictatorship.

Outline of the Proof

Lemma (decisiveness lemma)

A social choice function with at least three candidates that satisfies Pareto and independence must be decisive.

Lemma

Suppose a social choice function with at least three candidates satisfies Pareto and independence. Suppose there are two profiles in which no voter changes their mind about whether candidate A is preferred to candidate B. If A wins in the first profile, then B cannot win in the second profile.

Discussion Question

How is this different from independence alone?

Outline of the Proof

- Suppose we have a social choice function with more than three candidates that is Pareto and independent.
- It must be decisive.
- It satisfies the "super independence" property.

Claim

For each pair of candidates, there is some single voter who can force one candidate to lose by ranking the other candidate higher.

Claim

This voter must be the same for each pair of candidates.

The Decisiveness Lemma

Lemma (decisiveness lemma)

A social choice function with at least three candidates that satisfies Pareto and independence must be decisive.

Proof.

- Suppose we have a profile where A and B both win
- We will build a profile in which no candidate can win
- That's not possible, so we can't have a profile like that.

The Decisiveness Lemma

Proof.

- Suppose in some profile A and B both win
- Suppose x voters prefer A to B, and y voters prefer B to A
- $x \neq 0$ and $y \neq 0$:
- If $x = 0$ then all voters prefer B to A, and by Pareto we know A can't win.

The Decisiveness Lemma

Proof.

x	y
⋮	⋮
A	B
⋮	⋮
B	A
⋮	⋮

Profile P

New profile:

x	y
A	C
C	B
B	A
⋮	⋮

Profile Q

- We claim C is the unique winner in profile Q
- A can't win, by independence:
  - If A wins in Q, then A wins and B loses
  - Relative positions unchanged from Q to P
  - Then B can't win in P, contradicting our assumption
- B can't win, by Pareto
- Neither can anyone else

The Decisiveness Lemma

Proof.

x	y
⋮	⋮
A	B
⋮	⋮
B	A
⋮	⋮

Profile P

x	y
A	C
C	B
B	A
⋮	⋮

Profile Q

x	y
A	B
B	C
C	A
⋮	⋮

Profile R

- A loses by independence:
  - C wins and A loses in Q
  - Relative positions unchanged from Q to R
- B loses by independence:
  - Relative positions unchanged from P to R
  - A wins in P but loses in R
- No one else can win, by Pareto
- No one can win!

The Decisiveness Lemma

Lemma (decisiveness lemma)

A social choice function with at least three candidates that satisfies Pareto and independence must be decisive.

Proof.

- Suppose we have a profile where A and B both win
- We built a profile in which no candidate can win
- That's not possible, so we can't have a profile like that
- That proves the lemma.

□

A Corollary to Decisiveness

Lemma (decisiveness lemma)

A social choice function with at least three candidates that satisfies Pareto and independence must be decisive.

Corollary

It is impossible for a method to satisfy Pareto, independence, anonymity, and neutrality.

Proof.

- A Pareto and independent method must be decisive
- An anonymous and neutral method cannot be decisive
- We have to lose one.

□

Another Corollary

Lemma

Suppose a social choice function with at least three candidates satisfies Pareto and independence. Suppose there are two profiles in which no voter changes their mind about whether candidate A is preferred to candidate B. If A wins in the first profile, then B cannot win in the second profile.

Proof.

- Suppose A wins in the first profile
- Method is decisive by decisiveness lemma, so B can't win in first profile
- By independence, B can't win in second profile.

□

A New Idea: Dictatorial Control

- Want to show that a method has to be dictatorship
- Define a sort of part-way dictatorship

Definition

Suppose A and B are two candidates. We say a voter has dictatorial control for A over B if, whenever that voter prefers A to B, it will always be the case that B loses.

- Not necessarily symmetric!

Dictatorial Control

Lemma

If a social choice function with at least three candidates satisfies both Pareto and independence, then for any pair of candidates A and B, there is a voter with dictatorial control for A over B.

Outline of Proof.

- Switch preferences one at a time
- One specific voter decides between A and C
- That voter must have control for A over B by repeated independence arguments

□

Dictatorial Control

Claim

Some voter has dictatorial control for A over B

Proof.

- Suppose all voters rank C first, A second, B third
- C has to win by Pareto
- One by one, switch to A then B then C
- At the end, A wins by Pareto
- Some specific voter causes the winner to switch.

Dictatorial Control

Claim

Some voter has dictatorial control for A over B.

Proof.

X	v	Y
C	C	A
A	A	B
B	B	C
⋮	⋮	⋮

X	v	Y
C	A	A
A	B	B
B	C	C
⋮	⋮	⋮

- When v switches, that causes A to in, instead of C.
- Claim that v has dictatorial control for A over B.

Dictatorial Control

Claim

The voter v has dictatorial control for A over B.

Proof.

- Imagine profile P where v ranks A over B.
- Claim B can't win in P.
- Construct new profile Q:
  - Every voter in X puts C first, then A and B in same order as P.
  - Every voter in Y puts A and B first and second, in same order as P, then C third.
  - Voter v ranks A, then B, then C.

Dictatorial Control

Claim

The voter v has dictatorial control for A over B.

Proof.

Profile Q:

X		v	Y	
C	C	A	A	B
A	B	C	B	A
B	A	B	C	C
⋮	⋮	⋮	⋮	⋮

Dictatorial Control

Proof.

Profile 1

X	v	Y
C	C	A
A	A	B
B	B	C
⋮	⋮	⋮

Profile 2

X	v	Y
C	A	A
A	B	B
B	C	C
⋮	⋮	⋮

Profile Q

X		v	Y	
C	C	A	A	B
A	B	C	B	A
B	A	B	C	C
⋮	⋮	⋮	⋮	⋮

- C wins in profile 1 so by independence, B loses in profile Q.
- A wins in profile 2 so by independence, C loses in profile Q.
- By Pareto, only A can win in profile Q.
- By independence again, B can't win in original profile P.

Dictatorial Control

Claim

Some voter has dictatorial control for A over B.

Proof.

- Construct profiles where v is pivotal voter between A and C.
- Assume profile P where A is ranked over B.
- Build new profile Q where A must win.
- Independence shows that B can't win original profile P.

□

Lemma

If a social choice function with at least three candidates satisfies both Pareto and independence, then for any pair of candidates A and B, there is a voter with dictatorial control for A over B

Arrow's Theorem

Theorem (Arrow)

If a social choice function with at least three candidates satisfies both Pareto and independence, then it must be a dictatorship.

Proof.

- By decisiveness lemma, must be decisive
- By other lemma, for any pair of candidates A and B, some voter v has dictatorial control for A over B.
- Claim: v also has dictatorial control for B over A.
- Claim: Same voter has dictatorial control over *every* pair
- Thus method is dictatorship.

Arrow's Theorem

Claim

If a social choice function with at least three candidates satisfies both Pareto and independence, then if v has dictatorial control for A over B, they also have dictatorial control for B over A.

Proof.

v	w	Others
A	B	A
B	A	B
⋮	⋮	⋮

- Suppose w has control for B over A.
- B can't win because of v
- A can't win because of w
- No one else can win by Pareto
- No winner. Contradiction!
- So v must have control for B over A.

□

Arrow's Theorem

Claim

The same voter has dictatorial control over every pair .

Proof.

v	w	Others
C	B	C
A	C	B
B	A	A
⋮	⋮	⋮

- Suppose v has control between A and B, and w between B and C.
- B can't win because of v
- C can't win because of w
- No one else can win, by Pareto
- No winner. Contradiction!
- Same voter must have control over every pair.

Arrow's Theorem

Theorem (Arrow)

If a social choice function with at least three candidates satisfies both Pareto and independence, then it must be a dictatorship.

Proof.

- By decisiveness lemma, must be decisive
- By other lemma, for any pair of candidates A and B, some voter v has dictatorial control for A over B.
- v also has dictatorial control for B over A.
- v has dictatorial control over *every* pair
- Whichever candidate v ranks first wins
- The method is a dictatorship.

Tradeoffs

What have we learned?

- Can't be Condorcet and independent
- Can't be anonymous, neutral, and decisive
- Can't be Pareto, independent, and non-dictatorial.
- Can't hit half our criteria without being complicated!

Discussion Question

- What method would you want to use in an ideal world?
- What method should we use in *our* world?

Test on Monday

Test Rules

- Plan to take the whole class
- Will have 6-7 questions
- Bring a one-sided handwritten note sheet
- You can bring a calculator but it probably won't be useful

Test Topics

- Two-candidate methods and criteria
- Multi-candidate methods and criteria
- Short proofs and counter-examples
- Drawing conclusions from criteria