# Math 1231 Fall 2026
# Single-Variable Calculus I Section 12
# Mastery Quiz Worksheet 10
# Due Friday, October 30

This week's mastery quiz has three topics. It is the first time we will see topic S8 and the last quiz with topic S7. The quiz on Friday will have four questions on three topics, and should take about twenty minutes.

Remember that you are trying to demonstrate that you understand the concepts involved. For all these problems, you should justify your answers and show your work. Do not just write "yes" or "no" or give a single number.

On Friday, we will end class with a short quiz, containing questions drawn from this worksheet, plus one "easy" question on topic M1. You may not use or consult any materials or notes during Friday's quiz; make sure you work through these problems carefully, and understand them well enough that you can replicate that work in class.

**Topics on This Quiz**

- Major Topic 3: Optimization and Extrema
- Secondary Topic 7: Curve Sketching
- Secondary Topic 8: Physical Optimization

# Major Topic 3: Opitmization and Extrema

(a) The function $g(x) = (x^2 - 3x)\sqrt[3]{x-3}$ has absolute extrema either on $(-4,-1)$ or on $[1,4]$. Pick one of those intervals, explain why $g$ has extrema on that interval, and find the absolute extrema.

**Solution:** We compute

$$g'(x) = (2x-3)\sqrt[3]{x-3} + (x^2-3x)\frac{1}{3}(x-3)^{-2/3}.$$

This is undefined when $x = 3$, and to find the zeroes we compute

$$\begin{aligned}
0 &= (2x-3)\sqrt[3]{x-3} + (x^2-3x)\frac{1}{3}(x-3)^{-2/3} \\
&= (2x-3)\sqrt[3]{x-3} + \frac{x^2-3x}{3(x-3)^{2/3}} \\
0 &= (2x-3)\sqrt[3]{x-3}\cdot 3(x-3)^{2/3} + (x^2-3x) \\
&= (2x-3)3(x-3) + (x^2-3x) = 3(2x^2-3x-6x+9) + x^2 - 3x \\
&= 7x^2 - 30x + 27 = (7x-9)(x-3)
\end{aligned}$$

which is zero when $x = 3$ or $x = 9/7$. So then we have

$$\begin{aligned}
g(1) &= -2\sqrt[3]{-2} = 2\sqrt[3]{2} \\
g(9/7) &= \left(\frac{81}{49} - \frac{27}{7}\right)\sqrt[3]{-12/7} = \frac{108}{49}\sqrt[3]{12/7} \\
g(3) &= 0 \\
g(4) &= 4\cdot\sqrt[3]{1} = 4.
\end{aligned}$$

All of these numbers are non-negative, so the minimum value is 0, at $x = 3$. We can see that $2\sqrt[3]{2} < 2\cdot 2 = 4$, and maybe convince ourselves that

$$\frac{108}{49}\sqrt[3]{12/7} \approx 2\cdot 1 < 4.$$

Thus the maximum value is 4, which occurs at 4.

(b) Classify the critical points and relative extrema of $h(x) = \sin(x) + \cos(x)$ on $[0, 2\pi]$.

**Solution:** We have

$$h'(x) = \cos(x) - \sin(x)$$

so $h'(x)$ is defined everywhere, and has critical points where $\cos(x) = \sin(x)$. This happens when $x = \pi/4, 5\pi/4, 9\pi/4, \cdots = \pi/4 + n\pi$. We only need to care about $\pi/4$ and $5\pi/4$.

We can classify these points in two ways. We can use the first derivative test or the second derivative test. In these solutions I'll do both.

For the second derivative test we compute:

$$\begin{aligned} h''(x) &= -\sin(x) - \cos(x) \\ h''(\pi/4) &= -\sqrt{2}/2 - \sqrt{2}/2 = -\sqrt{2} < 0 \\ h''(5\pi/4) &= \sqrt{2}/2 + \sqrt{2}/2 = \sqrt{2} > 0. \end{aligned}$$

Thus $h$ has a local maximum at $\pi/4$ and has a local minimum at $5\pi/4$.

For the first derivative test we make a chart:

| | $h'(x)$ |
|---|---|
| $0 < x < \pi/4$ | $+$ |
| $\pi/4 < x < 5\pi/4$ | $-$ |
| $5\pi/4 < x < 2\pi$ | $+$ |

so $h$ has a relative maximum at $\pi/4$ and a relative minimum at $5\pi/4$.

# Secondary Topic 7: Curve Sketching

Let $g(x) = \dfrac{x^2 - 7}{x^2 - 4}$.

We can compute that $g'(x) = \dfrac{6x}{(x+2)^2(x-2)^2}$ and $g''(x) = \dfrac{-6(3x^2+4)}{(x^2-4)^3}$.

Sketch a graph of the function $g(x)$. Your answer should discuss the domain, asymptotes (including computing limits), roots, limits at infinity, critical points and values, intervals of increase and decrease, points of inflection, and concavity.

**Solution:** $g$ has domain all reals except $\pm 2$. $g(x) = 0$ when $x = \pm\sqrt{7}$. $\lim_{x\to\pm\infty} \frac{x^2-7}{x^2-4} = \lim_{x\to\pm\infty} \frac{1-7/x^2}{1-4/x^2} = 1$.

The derivative $g'(x) = \frac{6x}{(x+2)^2(x-2)^2}$ is defined except at $x = \pm 2$. It is zero when $x = 0$. The critical points are thus 0 and arguably $\pm 2$; $g(0) = 7/4$.

Since the denominator is a square, it is positive when $x > 0$ and negative when $x < 0$, so it is decreasing for negative $x$ and increasing for positive $x$. This means that $(0, 7/4)$ is a relative minimum.

We know that

$$g''(x) = \frac{-6(3x^2+4)}{(x^2-4)^3}$$

The numerator is negative everywhere. The denominator is negative when $-2 < x < 2$ and is positive when $x < -2$ or when $2 < x$. Thus the concavity only changes at $\pm 2$; it is concave up on $(-2, 2)$ and concave down elsewhere.

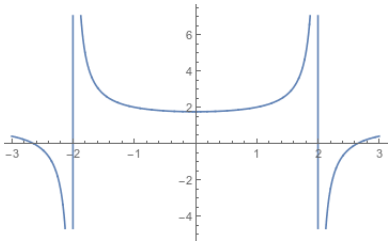


Figure 0.1: The graph of $g$ from $-3$ to 3

# Secondary Topic 8: Physical Optimization

(a) Find the point on the line $y = 2x + 5$ that is closest to the origin.

**Solution:** Our objective function is $D = \sqrt{x^2 + y^2}$, and our constraint is that $y = 2x + 3$. So we have

$$\begin{aligned} D(x) &= \sqrt{x^2 + (2x+5)^2} = \sqrt{x^2 + 4x^2 + 20x + 25} \\ &= \sqrt{5x^2 + 20x + 25} \\ D'(x) &= \frac{10x + 20}{2\sqrt{5x^2 + 20x + 25}} \end{aligned}$$

has a critical point only at $x = -2$, which gives us the point $(-2, 1)$.

To check this is really a minimum, we can't really use the EVT since $x$ can be infinitely big or small on the line. However, we can observe that the denominator of $D'(x)$ is

always positive, but the numerator is positive for $x > -2$ and negative for $x < -2$, which makes it a minimum. Or if we really want to, we can compute the second derivative

$$\begin{aligned} D''(x) &= \frac{20\sqrt{5x^2+20x+25} - (10x+20)\frac{10x+20}{\sqrt{5x^2+20x+25}}}{4(5x^2+20x+25)} \\ &= \frac{20(5x^2+20x+25) - (10x+20)^2}{4(5x^2+20x+25)^{3/2}} \\ &= \frac{100x^2+400x+500-100x^2-400x-400}{4(5x^2+20x+25)^{3/2}} \\ &= \frac{100}{4(5x^2+20x+25)^{3/2}} \end{aligned}$$

which is always positive. So the distance function is concave up, so we have a minimum.

(b) Suppose that a company that produces and sells $x$ units of a product each day makes a revenue of $R(x) = 260x - 9x^2/10$ dollars per day and has costs given by $C(x) = 1000 + 100x + x^2/10$ dollars per day. What is the maximum profit that can be made (where profit is revenues minus costs)?

**Solution:** Our profit function is $P(x) = R(x) - C(x) = -1000 + 160x - x^2$. Then

$$\begin{aligned} P'(x)160 - 2x \\ 160 = 2x \\ 80 = x \end{aligned}$$

We can check that this is truly a maximum by the second derivative: $P''(x) = -2 < 0$ so we have a local maximum.

Or we can see that $P'(x) > 0$ when $x < 80$ and $P'(x) < 0$ when $x > 80$, so the function is increasing until 80 and decreasing after.

The profit at this quantity is

$$P(80) = -1000 + 160(80) - (80)^2 = -1000 + 12800 - 6400 = 5400.$$

Thus our maximum profit is 5400 dollars per day.