Math 1231 Fall 2026
Single-Variable Calculus I Section 12
Mastery Quiz Worksheet 11
Due Friday, November 6

This week's mastery quiz has three topics. It is the last time we will see topic S8 and the first time we see topic S9. The quiz on Friday will have four questions on three topics, and should take about 25 minutes.

Remember that you are trying to demonstrate that you understand the concepts involved. For all these problems, you should justify your answers and show your work. Do not just write "yes" or "no" or give a single number.

On Friday, we will end class with a short quiz, containing questions drawn from this worksheet, plus one "easy" question on topic M1. You may not use or consult any materials or notes during Friday's quiz; make sure you work through these problems carefully, and understand them well enough that you can replicate that work in class.

Topics on This Quiz

- Major Topic 3: Optimization and Extrema
- Secondary Topic 8: Physical Optimization
- Secondary Topic 9: Riemann Sums

Major Topic 3: Opitmization and Extrema

(a) The function $f(x) = \dfrac{x^3 - 5x^2}{x+3}$ has absolute extrema either on the interval $[-4,-1]$ or on the interval $[-1,4]$. Pick one of those intervals, explain why f has extrema on that interval, and find the absolute extrema.

Solution: f is continuous on the closed interval $[-1,4]$, so it must have extrema there. (It is *not* continuous on $[-4,-1]$ because it is undefined at -3.)

We compute

$$\begin{aligned} f'(x) &= \frac{(3x^2-10x)(x+3) - (x^3-5x^2)\cdot 1}{(x+3)^2} \\ &= \frac{3x^3+9x^2-10x^2-30x-x^3+5x^2}{(x+3)^2} \\ &= \frac{2x^3+4x^2-30x}{(x+3)^2} = \frac{(2x)(x^2+2x-15)}{(x+3)^2} \\ &= \frac{(2x)(x+5)(x-3)}{(x+3)^2} \end{aligned}$$

is zero for $x = 0, 3, -5$ and is undefined for $x = -3$, so those are the critical points. The only ones we have to care about are $x = 0, 3$.

$$\begin{aligned} f(-1) &= -3 \\ f(0) &= 0 \\ f(3) &= -3 \\ f(4) &= -16/7 \end{aligned}$$

so the absolute minimum is -3 at -1 and 3, and the absolute maximum is 0 at 0.

(b) Find and classify the critical points of $f(x) = \sqrt[3]{x^3 - 3x}$.

Solution: We have

$$f'(x) = \frac{1}{3}(x^3-3x)^{-2/3}(3x^2-3) = \frac{(x-1)(x+1)}{\sqrt[3]{x(x^2-3)}^2}.$$

This is zero when $x = \pm 1$ and is undefined when $x = 0$ or $x = \pm\sqrt{3}$. Thus the critical points are $-\sqrt{3}, -1, 0, 1, \sqrt{3}$.

We could try the second derivative test, but we know it won't work at $\pm\sqrt{3}$ or at 0, since those are non-differentiable critical points. So we make a chart:

	$(x-1)$	$(x+1)$	$\sqrt[3]{x^2}$	$\sqrt[3]{(x^2-3)^2}$	f'
$x < -\sqrt{3}$	$-$	$-$	$+$	$+$	$+$
$-\sqrt{3} < x < -1$	$-$	$-$	$+$	$+$	$+$
$-1 < x < 0$	$-$	$+$	$+$	$+$	$-$
$0 < x < 1$	$-$	$+$	$+$	$+$	$-$
$1 < x < \sqrt{3}$	$+$	$+$	$+$	$+$	$+$
$\sqrt{3} < x$	$+$	$+$	$+$	$+$	$+$

So we see the function has a local maximum at -1 and a local minimum at 1; the critical points at $-\sqrt{3}, 0$, and $\sqrt{3}$ are neither minima nor maxima.

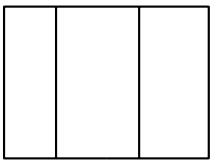
Secondary Topic 8: Physical Optimization

(a) We wish to build a rectangular pen with two parallel internal partitions, using 1000 feet of fencing. We want to maximize the total area of the pen.

(a) What is your objective function, and why?

(b) What constraint equation(s) can you use?

(c) What dimensions maximize the total area of the pen?



Solution: Our objective function is $A = \ell w$, because this is the area we want to maximize. We see also that we have the constraint $2\ell + 4w = 1000$ so we can write $\ell = 500 - 2w$, and thus we have

$$A = (500 - 2w)w = 500w - 2w^2$$
$$A' = 500 - 4w$$

has a critical point when $w = 125$.

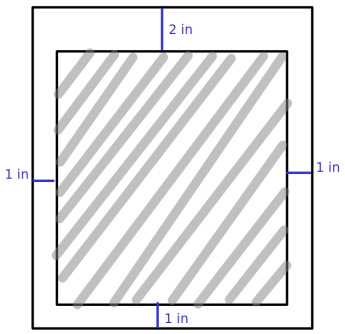
We can see this is a maximum using the extreme value theorem: the function is defined on the interval $[0, 250]$, and $A(0) = A(250) = 0$.

Or we can use the first derivative test; we see that $A'(w) < 0$ when $w > 125$ and $A'(w) > 0$ when $w < 125$, so A has a local maximum at $w = 125$.

Or we can use the second derivative test. $A'' = -4 < 0$ so we have a local maximum.

Thus the pen is maximized with width 125 and length 250. (The maximum area, which I didn't ask for, is 31250 square feet.)

(b) A poster needs to have an area of 216in^2, with 1-inch margins on the bottom and sides and a 2-inch margin on the top. What dimension maximize the area of the *printed* region, excluding margins?



Solution: Our constraint is that $h \cdot w = 216$, and we want to maximize $(h-3)(w-2)$. So we have $h = 216/w$, and thus we want to maximize

$$A(w) = \left(\frac{216}{w} - 3\right)(w-2) = 180 - 3w - \frac{432}{w} + 6$$
$$A'(w) = -3 + \frac{432}{w^2}$$
$$w^2 = 144$$
$$w = \pm 12.$$

Obviously the only relevant critical point is the positive one, 12.

We can check that this is truly a maximum by the second derivative: $A''(w) = \frac{-720}{w^3} < 0$ for any $w > 0$, so we have a local maximum.

Or we can see that $A'(x) > 0$ when w is small and $A'(w) < 0$ when w is large, so the function is increasing until 80 and decreasing after.

So the dimensions we want are a poster with width 12 inches and height $216/12 = 18$ inches.

Secondary Topic 9: Riemann Sums

Let $f(x) = x^2 - x$ be defined on the interval $[-3, 0]$.

(a) Approximate the area under the curve of the function using three rectangles and right endpoints.

(b) Approximate the area under the curve of the function using three rectangles and left endpoints.

(c) Write a formula for R_n, the estimate using n rectangles and right endpoints, as a summation of n terms.

(d) Use your answer in part (c) to find a closed-form formula for R_n. (This formula should not have a summation sign or be given as a sum of n terms.)

(e) Use the formula in part (c) to compute the area exactly.

Solution:

(a) $R_3 = 1 \cdot f(-2) + 1 \cdot f(-1) + 1 \cdot f(0) = 6 + 2 + 0 = 8$.

(b) $L_3 = 1 \cdot f(-3) + 1 \cdot f(-2) + 1 \cdot f(-1) = 12 + 6 + 2 = 20$.

(c)
$$R_n = \sum_{i=1}^{n} \frac{3}{n} f\left(-3 + i\frac{3}{n}\right) = \sum_{i=1}^{n} \frac{3}{n}\left((3i/n - 3)^2 - (3i/n - 3)\right)$$

(d)

$$\begin{aligned}
R_n &= \sum_{i=1}^{n} \frac{3}{n} f\left(-3 + i\frac{3}{n}\right) = \sum_{i=1}^{n} \frac{3}{n}\left((3i/n - 3)^2 - (3i/n - 3)\right) \\
&= \sum_{i=1}^{n} \frac{3}{n}\left(\frac{9i^2}{n^2} - \frac{18i}{n} + 9 - \frac{3i}{n} + 3\right) \\
&= \sum_{i=1}^{n} \frac{27i^2}{n^3} - \frac{63i}{n^2} + \frac{36}{n} \\
&= \frac{27}{n^3}\sum_{i=1}^{n} i^2 - \frac{63}{n^2}\sum_{i=1}^{n} i + \frac{36}{n}\sum_{i=1}^{n} 1 \\
&= \frac{27}{n^3}\frac{n(n+1)(2n+1)}{6} - \frac{63}{n^2}\frac{n(n+1)}{2} + \frac{36}{n}n
\end{aligned}$$

(e) We can compute

$$\begin{aligned}
\lim_{n\to+\infty} R_n &= \lim_{n\to+\infty} \frac{27}{n^3}\frac{n(n+1)(2n+1)}{6} - \frac{63}{n^2}\frac{n(n+1)}{2} + \frac{36}{n}n \\
&= \lim_{n\to+\infty} \frac{27 \cdot 1(1+1/n)(2+1/n)}{6} - \frac{63 \cdot 1(1+1/n)}{2} + 36 \\
&= 9 - \frac{63}{2} + 36 = \frac{27}{2}.
\end{aligned}$$