

Math 1231: Single-Variable Calculus I
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Recitation 3

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Problem 1. Let $f(x) = \frac{3}{x-3} - \frac{x+6}{x^2-3x}$.

- (a) Is this function continuous where it's defined? Where is it undefined?
- (b) We can't really "factor" this, because it's a sum of two terms. Instead we need to collect terms together so we can factor it. How can we collect the terms and factor the expression?
- (c) What is $\lim_{x \rightarrow 3} f(x)$?

Solution:

- (a) The function is reasonable, so it's continuous where defined. It's undefined at $x = 0$ and $x = 3$.
- (b) We need to find a common denominator so we can subtract the fractions. We can compute

$$\begin{aligned} \frac{3}{x-3} - \frac{x+6}{x^2-3x} &= \frac{3(x^2-3x)}{(x-3)(x^2-3x)} - \frac{(x+6)(x-3)}{(x^2-3x)(x-3)} \\ &= \frac{3x^2-9x-(x^2+3x-18)}{(x-3)(x^2-3x)} \\ &= \frac{2x^2-12x+18}{(x-3)(x^2-3x)} \\ &= \frac{2(x-3)^2}{x(x-3)^2}. \end{aligned}$$

This function is not the same as $\frac{2}{x}$, but almost the same.

There's actually an easier way to handle the algebra, though. We could have started by factoring the second fraction, and gotten

$$\frac{3}{x-3} - \frac{x+6}{x^2-3x} = \frac{3}{x-3} - \frac{x+6}{x(x-3)}.$$

Then we see we just need to multiply the first term on the top and bottom by x :

$$\begin{aligned} \frac{3}{x-3} - \frac{x+6}{x^2-3x} &= \frac{3}{x-3} - \frac{x+6}{x(x-3)} \\ &= \frac{3x}{x(x-3)} - \frac{x+6}{x(x-3)} \\ &= \frac{3x - (x+6)}{x(x-3)} \\ &= \frac{2x-6}{x(x-3)} = \frac{2(x-3)}{x(x-3)}. \end{aligned}$$

(c)

$$\begin{aligned} \lim_{x \rightarrow 3} f(x) &= \lim_{x \rightarrow 3} \frac{3}{x-3} - \frac{x+6}{x^2-3x} \\ &= \lim_{x \rightarrow 3} \frac{2(x-3)}{x(x-3)} \\ &= \lim_{x \rightarrow 3} \frac{2}{x} = \frac{2}{3}. \end{aligned}$$

Problem 2. We want to compute $\lim_{x \rightarrow 3} \frac{\sin(x^2-9)}{x-3}$.

- (a) What rule do we know we need to invoke here?
- (b) What θ are we going to need to pick for this to work out, and why?
- (c) Do algebra so that you can invoke the small angle approximation. What is the limit? (Are you using the AIF property?)
- (d) Go back to the beginning, and see what our heuristic idea that $\sin(\theta) \approx \theta$ would have told you. Does that match with what you got?

Solution:

- (a) We need to use the small angle approximation, because this problem requires us to make trig interact with algebra.

(b) We basically have to take $\theta = x^2 - 9$, because that's what's inside the sin.

(c) We need to get a θ on the bottom as well. So we take

$$\begin{aligned}\lim_{x \rightarrow 3} \frac{\sin(x^2 - 9)}{x - 3} &=_{AIF} \lim_{x \rightarrow 3} \frac{\sin(x^2 - 9)}{x - 3} \cdot \frac{x + 3}{x + 3} = \lim_{x \rightarrow 3} \frac{\sin(x^2 - 9)(x + 3)}{x^2 - 9} \\ &= \lim_{x \rightarrow 3} \frac{\sin(x^2 - 9)}{x^2 - 9} \cdot \lim_{x \rightarrow 3} x + 3 = 1 \cdot (3 + 3) = 6.\end{aligned}$$

You have to use AIF in that first equality, because you're making the function undefined at -3 .

There are a couple different ways to make this algebra work out. You might also try

$$\begin{aligned}\lim_{x \rightarrow 3} \frac{\sin(x^2 - 9)}{x - 3} &=_{AIF} \lim_{x \rightarrow 3} \frac{\sin(x^2 - 9)/x^2 - 9 \cdot x^2 - 9}{x - 3} \\ &= \lim_{x \rightarrow 3} \frac{x^2 - 9}{x - 3} =_{AIF} \lim_{x \rightarrow 3} x + 3 = 6,\end{aligned}$$

which is essentially the same logic.

(d)

We have a $\sin(0)$ on the top and a 0 on the bottom, but the 0 s don't come from the same form; we need to get a $x^2 - 9$ term on the bottom. Multiplication by the conjugate gives

$$\begin{aligned}\lim_{x \rightarrow 3} \frac{\sin(x^2 - 9)}{x - 3} &= \lim_{x \rightarrow 3} \frac{\sin(x^2 - 9)}{x - 3} \cdot \frac{x + 3}{x + 3} = \lim_{x \rightarrow 3} \frac{\sin(x^2 - 9)(x + 3)}{x^2 - 9} \\ &= \lim_{x \rightarrow 3} \frac{\sin(x^2 - 9)}{x^2 - 9} \cdot \lim_{x \rightarrow 3} x + 3 = 1 \cdot (3 + 3) = 6.\end{aligned}$$

Problem 3. We want to think about the ways that infinity doesn't really work like a number, and we can't do arithmetic with it.

(a) To start: what is $\lim_{x \rightarrow 0} 1/x$, and why?

(b) Let's look at $\lim_{x \rightarrow 0} 1/x + 1/x$. If we computed the limit of each fraction individually, what indeterminate form would we get?

(c) How do we actually compute $\lim_{x \rightarrow 0} \frac{1}{x} + \frac{1}{x}$? (Hint: combine them into one fraction.) Does this make sense in light of what you got in part (b)?

(d) Now consider $\lim_{x \rightarrow 0} \frac{1}{x} + \frac{x-1}{x-x^2}$. What is the limit of each piece, and what indeterminate form is this?

- (e) Compute $\lim_{x \rightarrow 0} \frac{1}{x} + \frac{x-1}{x-x^2}$ directly. Does this make sense in light of what you got in part (d)?
- (f) Now consider $\lim_{x \rightarrow 0} 1/x + 1/x^2$. What indeterminate form would this represent? What is the limit? Do those make sense together?
- (g) Finally, let's look at $\lim_{x \rightarrow 0} \frac{1}{x} + \frac{x^2-3x+2}{x^2-2x}$. What indeterminate form is this? What is the limit?
- (h) What pattern do you see from all of these?

Solution:

- (a) $\lim_{x \rightarrow 0} \frac{1 \nearrow 1}{x \searrow 0} = \pm\infty$.
- (b) This looks like $\infty + \infty$ as an indeterminate form.
- (c) We see $\lim_{x \rightarrow 0} \frac{1}{x} + \frac{1}{x} = \lim_{x \rightarrow 0} \frac{2 \nearrow 2}{x \searrow 0} = \pm\infty$. This seems to make sense; $\infty + \infty = \infty$ is perfectly reasonable.
- (d) We already know $\lim_{x \rightarrow 0} \frac{1}{x} = \pm\infty$. We can compute that

$$\lim_{x \rightarrow 0} \frac{x-1 \nearrow -1}{x-x^2 \searrow 0} = \pm\infty.$$

So this is again $\infty + \infty$.

- (e) By combining fractions, we get

$$\lim_{x \rightarrow 0} \frac{1}{x} + \frac{x-1}{x-x^2} = \lim_{x \rightarrow 0} \frac{1-x}{x-x^2} + \frac{x-1}{x-x^2} = \lim_{x \rightarrow 0} 0 = 0.$$

So here $\infty + \infty = 0$.

- (f) We have a $\pm\infty$ plus a $+\infty$, so we get $\infty + \infty$ again. When we combine them into one term we get

$$\lim_{x \rightarrow 0} \frac{1}{x} + \frac{1}{x^2} = \lim_{x \rightarrow 0} \frac{x+1 \nearrow 1}{x^2 \searrow 0} = +\infty$$

since the denominator is $x^2 \geq 0$. So here $\infty + \infty = +\infty$.

We could heuristically say that $\frac{1}{x^2}$ goes to $+\infty$ “faster” than $\frac{1}{x}$ goes to $\pm\infty$, and so it wins out; but this is really vague and handwavy so we try to replace it with more precise arguments like this one.

- (g) We compute $\lim_{x \rightarrow 0} \frac{x-3x+2}{x^2-2x} = \pm\infty$, so this is, again, $\infty + \infty$. The actual limit is

$$\lim_{x \rightarrow 0} \frac{1}{x} + \frac{x^2 - 3x + 2}{x^2 - 2x} = \lim_{x \rightarrow 0} \frac{x - 2 + x^2 - 3x + 2}{x^2 - 2x} = \lim_{x \rightarrow 0} \frac{x^2 - 2x}{x^2 - 2x} = \lim_{x \rightarrow 0} 1 = 1.$$

So here $\infty + \infty = 1$.

- (h) In conclusion, if you know something looks like $\infty + \infty$, you don't really know anything about it at all.

Problem 4. Let $f(x) = x^3$. We want to find a formula for the derivative of this function at any given point.

- Write down a formula for $f'(a)$ using the $h \rightarrow 0$ limit formulation. What does the numerator mean? What does the denominator mean?
- Use your formula from part (a) to compute the derivative.
- Now write down a formula for $f'(a)$ using the $x \rightarrow a$ limit formulation. Does this look easier or harder than the formula from part (a), and why? What does the numerator mean? What does the denominator mean?
- Use the formula from part (c) to compute the derivative. You should get the same answer you got in part (b).
- Which method was faster? Which method was easier?

Solution:

- (a)

$$f'(a) = \lim_{h \rightarrow 0} \frac{f(a+h) - f(a)}{h} = \lim_{h \rightarrow 0} \frac{(a+h)^3 - a^3}{h}$$

The top is the difference between two output values; the bottom is the difference between the corresponding inputs. You can think of the bottom as “change the input by a bit” and the top as the difference between the two outputs.

(b)

$$\begin{aligned}
 f'(a) &= \lim_{h \rightarrow 0} \frac{(a+h)^3 - a^3}{h} \\
 &= \lim_{h \rightarrow 0} \frac{a^3 + 3a^2h + 3ah^2 + h^3 - a^3}{h} \\
 &= \lim_{h \rightarrow 0} \frac{3a^2h + 3ah^2 + h^3}{h} \\
 &=^{AIF} \lim_{h \rightarrow 0} 3a^2 + 3ah + h^2 = 3a^2.
 \end{aligned}$$

(c)

$$f'(a) = \lim_{x \rightarrow a} \frac{f(x) - f(a)}{x - a} = \lim_{x \rightarrow a} \frac{x^3 - a^3}{x - a}$$

Again the top is the difference in outputs and the bottom is the difference in inputs. Here we can see two specific inputs, x and a , on the bottom; the top is the two corresponding outputs.

(d)

$$\begin{aligned}
 f'(a) &= \lim_{x \rightarrow a} \frac{x^3 - a^3}{x - a} \\
 &= \lim_{x \rightarrow a} \frac{(x - a)(x^2 + ax + a^2)}{x - a} \\
 &=^{AIF} \lim_{x \rightarrow a} x^2 + ax + a^2 = 3a^2.
 \end{aligned}$$

Notice we use the difference of cubes formula from section 1.1 of the notes.

(e) To my eyes, at least, the $h \rightarrow 0$ method is more straightforward, but the $x \rightarrow a$ method is faster *if you know the trick*. If you look at it and immediately see that $x^3 - a^3 = (x - a)(x^2 + ax + a^2)$, then the $x \rightarrow a$ method works very quickly. But if you don't know or remember that fact, it's hard to figure out what to do at all; you just get stuck.

In contrast, the $h \rightarrow 0$ method takes more algebra and work and writing and time, but less cleverness and thinking. If you just multiply everything out and cancel out the obvious stuff, it works out. When I don't know what I'm doing, I default to the $h \rightarrow 0$ version.