Math 1231 Fall 2026
Single-Variable Calculus I Section 12
Mastery Quiz Worksheet 7
Due Friday, October 9

This week's mastery quiz has four topics. It is the last time we will topic S4 or S5, and the first time we have seen topic S6.

Remember that you are trying to demonstrate that you understand the concepts involved. For all these problems, you should justify your answers and show your work. Do not just write "yes" or "no" or give a single number.

On Friday, we will end class with a short quiz, containing questions drawn from this worksheet, plus one "easy" question on topic M1. You may not use or consult any materials or notes during Friday's quiz; make sure you work through these problems carefully, and understand them well enough that you can replicate that work in class.

Topics on This Quiz

- Major Topic 2: Computing Derivatives
- Secondary Topic 4: Rates of Change
- Secondary Topic 5: Implicit Differentiation
- Secondary Topic 6: Related Rates

Major Topic 2: Computing Derivatives

(a) Compute $\dfrac{d}{dx}\dfrac{\sin(\csc(x^2+1))}{x^4+\cos(x)} =$

Solution:

$$\frac{\big(\cos(\csc(x^2+1))(-\csc(x^2+1)\cot(x^2+1))2x\big)(x^4+\cos(x)) - (4x^3-\sin(x))\sin(\csc(x^2+1))}{(x^4+\cos(x))^2}.$$

(b) Compute $\dfrac{d}{dx}\tan^{3/5}\left(\sec\left(x^3-4\right)\right) =$

Solution:

$$\frac{3}{5}\tan^{-2/5}\left(\sec\left(x^3-4\right)\right)\sec^2\left(\sec\left(x^3-4\right)\right) \cdot\sec\left(x^3-4\right)\tan\left(x^3-4\right)3x^2$$

(c) Compute $\dfrac{d}{dx}\cos^2\left(\tan^2\left(\sec^2(\sqrt{x}+x)\right)\right) =$

Solution:

$$2\cos\left(\tan^2\left(\sec^2(\sqrt{x}+x)\right)\right)\left(-\sin\left(\tan^2\left(\sec^2(\sqrt{x}+x)\right)\right)\right) \cdot 2\tan\left(\sec^2(\sqrt{x}+x)\right)\sec^2\left(\sec^2(\sqrt{x}+x)\right) \cdot 2\sec(\sqrt{x}+x)\sec(\sqrt{x}+x)\tan(\sqrt{x}+x)\left(\frac{1}{2x}+1\right)$$

Secondary Topic 4: Rates of Change

(a) Suppose the vertical position of a weight on a spring in inches is given as a function of time in seconds by the formula $h(t) = \cos(2t)$.

(i) When is the velocity zero?

(ii) When is the acceleration zero?

Solution:

(i) $p'(t) = -2\sin(2t)$ so the velocity is zero when $\sin(2t) = 0$. This happens when $2t = 0, \pi, 2\pi, \ldots$, and thus when $t = 0, \pi/2, \pi, 3\pi/2, \ldots$. In other words, at $n\pi/2$.

(ii) $p''(t) = -4\cos(2t)$ is zero when $2t = \pi/2, 3\pi/2, \ldots$, and thus when $t = \pi/4, 3\pi/4, 5\pi/4, \ldots$. We could say $t = (2n+1)\pi/4$.

(b) Let $F(x) = \frac{1}{x} + 1$ be the amount of pressure exerted on a beam in pounds per square inch at a point x inches to the right of its left end.

(i) What are the units of $F'(x)$? What does $F'(x)$ represent physically? What would it mean if $F'(x)$ is big?

(ii) Compute $F'(5)$. What does this tell you physically? What physical observation could you make to check your calculation?

Solution:

(i) The derivative $F'(x)$ has units pounds per square inch per inch, or pounds per cubic inch. $F'(x)$ is the rate at which pressure is increasing as you move to the right along the stick. If $F'(x)$ is big, that means that moving along the stick a little bit will increase the pressure by a lot.

(ii) $F'(x) = -1/x^2$ so $F'(5) = -1/25$. This means that if we are five inches to the right of the endpoint, moving one more inch to the right should decrease the pressure by about $1/25$ of a pound per square inch.

Secondary Topic 5: Implicit Differentiation

(a) Find a formula for $\frac{d^2y}{dx^2}$ in terms of x and y if $\sin(xy) = x + y$.

Solution: Using implicit differentiation, we have

$$\cos(xy)(y + xy') = 1 + y'$$
$$y\cos(xy) - 1 = (1 - \cos(xy)x)y'$$

$$y' = (\frac{y\cos(xy) - 1}{1 - \cos(xy)x}$$

$$y'' = \frac{(y'\cos(xy) - y\sin(xy)(y + xy'))\,(1 - \cos(xy)x) - (y\cos(xy) - 1)(x\sin(xy)(y + xy') - \cos(xy))}{(1 - \cos(xy)x)^2}$$

$$= \frac{\left(\left(\frac{y\cos(xy)-1}{1-\cos(xy)x}\right)\cos(xy) - y\sin(xy)(y + x\left(\frac{y\cos(xy)-1}{1-\cos(xy)x}\right))\right)(1 - \cos(xy)x) - (y\cos(xy) - 1)(x\sin(xy}{(1 - \cos(xy)x)^2}$$

(b) Find an equation for the tangent line to the curve $x^2y^2 = 5 + x + y$ at the point $(1, 3)$.

Solution: Implicit differentiation gives us

$$\begin{aligned} 2xy^2 + 2x^2yy' &= 1 + y' \\ 2 \cdot 1 \cdot 9 + 2 \cdot 1^2 \cdot 3 \cdot y' &= 1 + y' \\ 18 + 6y' &= 1 + y' \\ 5y' &= -17 \\ y' &= -17/5 \end{aligned}$$

and thus the tangent line has equation

$$y - 3 = \frac{-17}{5}(x - 1).$$

Alternatively we can compute

$$\begin{aligned} 2xy^2 + 2x^2yy' &= 1 + y' \\ y'(2x^2y - 1) &= 1 - 2xy^2 \\ y' &= \frac{1 - 2xy^2}{2x^2y - 1} \end{aligned}$$

Secondary Topic 6: Related Rates

(a) A snowball is melting such that its surface area is decreasing at $1\text{cm}^2/\text{min}$. When the radius is 8cm, how quickly is the radius decreasing?

(The surface area of a sphere of radius r is $4\pi r^2$.)

(a) Choose an equation to use for this problem, and explain why you chose it.

(b) Use calculus to answer the question. Answer with a complete sentence.

Solution: We have $S = 4\pi r^2$, so $S' = 8\pi r r'$. When the radius is 8cm we have

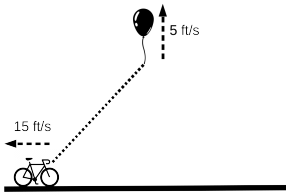
$$\begin{aligned} S' &= 8\pi \cdot 8\text{cm} \cdot r' \\ -1\text{cm}^2/\text{min} &= 64\pi r' \\ r' &= \frac{-1}{64\pi}\text{cm/min}. \end{aligned}$$

Thus when the radius is 8cm, the radius of the snowball is decreasing by $\frac{1}{64\pi}$ centimeters per minute.

(b) A balloon is rising at a constant speed of 5 feet per second. A boy is cycling along a straight road at a speed of 15 feet per second. When he passes under the balloon, it is 45 feet above him. We want to know how fast is he distance between the boy and the balloon is increasing 3 seconds later.

(a) Choose an equation to use for this problem, and explain why you chose it.

(b) Use calculus to answer the question. Answer with a complete sentence.



Solution: We know one distance, and how fast it's changing. We want to know how fast another distance is changing, so the Pythagorean theorem, which relates distances to each other, seems like a reasonable choice. So we use the equation $d^2 = w^2 + h^2$.

We see that the height of the balloon is $h = 60\text{ft}$, and the derivative is $h' = 5\text{ft/s}$. The distance between the boy and the point under the balloon is $w = 45\text{ft}$ and the derivative is $w' = 15\text{ft/s}$. The distance between them is given by $d^2 = w^2 + h^2$, and so we can compute first that the current distance is 75 feet, and then that

$$\begin{aligned} 2dd' &= 2ww' + 2hh' \\ dd' &= ww' + hh' \\ 75\text{ft}d' &= 45\text{ft} \cdot 15\text{ft/s} + 60\text{ft} \cdot 5\text{ft/s} = 675\text{ft}^2/\text{s} + 300\text{ft}^2/\text{s} = 975\text{ft}^2/\text{s} \\ d' &= \frac{975}{75}\text{ft/s} = \frac{325}{25}\text{ft/s} = 13\text{ft/s}. \end{aligned}$$

Thus the distance between the boy and the balloon is increasing by 13 feet per second.