Math 1231: Single-Variable Calculus I
George Washington University Fall 2026
Recitation 7

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October 14, 2025

Problem 1. Consider the function $f(x) = x^3 - 3x^2 + 1$ on $[-1, 4]$.

(a) Does this function have absolute extrema? Why?

(b) What are the critical points of this function?

(c) How many absolute extrema are there? What are they, and where are they?

Solution:

(a) This function is continuous on a closed interval, so by the Extreme Value Theorem it must have an absolute maximum and an absolute minimum.

(b) $f'(x) = 3x^2 - 6x$ and is defined everywhere. We have $3x^2 - 6x = 0$ when $x = 0$ or $x = 2$, so the critical points are 0 and 2.

(c) We need to evaluate all the possible extrema: the critical poitns and the endpoints. We compute $f(-1) = -3, f(0) = 1, f(2) = -3, f(4) = 17$. Thus the absolute maximum is 17 at 4, and the absolute minimum is -3 at -1 and 2.

Problem 2. Let's find the global extrema of $g(x) = \sqrt[3]{x^3 + 6x^2}$ on the closed interval $[-5, 5]$.

(a) Does this function have absolute extrema? Why?

(b) What are the critical points of this function?

(c) How many absolute extrema are there? What are they, and where are they? (Hint: you may need to use a calculator at the last step.)

Solution:

(a) This function is continuous on a closed interval, so by the Extreme Value Theorem it must have an absolute maximum and an absolute minimum.

(b) We take the derivative, and compute

$$g'(x) = \frac{1}{3}(x^3+6x^2)^{-2/3}(3x^2+12x) = \frac{3x(x+4)}{3\sqrt[3]{(x^3+6x^2)^2}}.$$

This derivative is zero when $x=-4$, and it does't exist when $x=0$ or $x=-6$. (You might think that $g'(0)=0$, but it's actually just undefined.) The critical points are $-6,-4,0$, but we can ignore the case where $x=-6$, since it's not inside the interval.

(c) We need to evaluate all the possible extrema: the critical points inside the interval, and the endpoints. We might need a calculator here, but we

$$\begin{aligned} g(-5) &= \sqrt[3]{-125+150} = \sqrt[3]{25} \approx 2.9 \\ g(-4) &= \sqrt[3]{-64+96} = \sqrt[3]{32} \approx 3.17 \\ g(0) &= \sqrt[3]{0} = 0 \\ g(5) &= \sqrt[3]{125+150} = \sqrt[3]{275} \approx 6.5. \end{aligned}$$

Thus the absolute minimum is 0, which occurs at 0, and the absolute maximum is $\sqrt[3]{275} \approx 6.5$, which occurs at 5.

Note that if you forget about the critical point where $g'(x)$ is undefined, you will miss the minimum! The minimum is very definitely not $\sqrt[3]{25}$, which you can see if you graph the function.

Problem 3. Let $f(x) = |x^3 - 3x|$ on the closed interval $[-1,3]$.

(a) Does this function have absolute extrema? Why?

(b) What are the critical points of this function?

(c) How many absolute extrema are there? What are they, and where are they?

Solution:

(a) This function is continuous on a closed interval, so by the Extreme Value Theorem it must have extrema.

(b) Absolute value is funny here. Technically speaking, we know that

$$f'(x) = \frac{x^3 - 3x}{|x^3 - 3x|} \cdot (3x^2 - 3).$$

This is undefined when $0 = x^3 - 3x = x(x^2 - 3) = x(x - \sqrt{3})(x + \sqrt{3})$, which happens when $x = 0$ or $x = \pm\sqrt{3}$. It's zero when $3x^2 - 3 = 0$, which happens when $x = \pm 1$.

(c) We only care what happens in $[-1, 3]$, so we care about the critical points $-1, 0, 1, \sqrt{3}$, and also the endpoints -1 and 3. (But of course we don't have to look at -1 twice). We compute

$$f(-1) = 2 \qquad f(0) = 0 \qquad f(1) = 2$$
$$f(\sqrt{3}) = 0 \qquad f(3) = 18.$$

Thus we see the maximum value is 18, achieved at 3; and the minimum value is 0, achieved at 0 and $\sqrt{3}$.

The non-differentiable points were very important here. If we forgot about them, we'd only see $-1, 1, 3$, and conclude that the minimum value was 2.

Problem 4. We can use the mean value theorem to constrain the possible values for a function. For instance, suppose I have a function f, and all I know is that $f(1) = 10$ and $f'(x) \geq 2$ for every x. I want to know about $f(4)$.

(a) What would this mean in English? How should we think about this physically? What does that tell you about $f(4)$?

(b) Use the Mean Value Theorem to set up an equation relating $f(1), f(4)$, and $f'(x)$. What does it tell you about $f(4)$?

(c) How do those two arguments relate to each other?

(d) is it possible for $f(4) = -30$?

Solution:

(a) Your location at time 1 is 10; maybe this is "your distance from home", so at 1 PM you're ten miles from home. Your speed is always at most 2, meaning that you're going no more that 2 miles per hour away from home. At 4 PM, you'll be at most 16 miles from home.

(b) f is continuous and differentiable, so there's some c such that

$$\frac{f(4)-f(1)}{4-1} = f'(c) < 2$$
$$f(4) - 10 < 2 \cdot 3$$
$$f(4) < 16.$$

(c) We did get the same answer both ways! The second argument is just a formalization of the first.

(d) Yes! The not-very-physical idea in this problem is that your speed is always less than 2, but it can be as *negative* as you want. We're essentially allowing infinite speed, but only backwards.

Problem 5. Suppose $|g'(x)| \leq 2$ for all x, and $g(0) = 7$. We want to know about $g(5)$?

(a) What would this mean in English? How should we think about this physically? What does that tell you about $g(5)$?

(b) Use the Mean Value Theorem to set up an equation relating $g(0), g(5)$, and $g'(x)$. What does it tell you about $f(4)$?

(c) How do those two arguments relate to each other?

(d) is it possible for $g(5) = -30$?

Solution:

(a) At time 0 (noon?) we're 7 miles from home. Our speed is never more than 2 miles per hour, so over five hours we'll travel at most ten miles. We could get further from home and be as much as 17 miles away. Or we can go the other direction, overshoot, and be -3 miles away.

(b) g is continuous and differentiable, so there's some c such that

$$\frac{g(5)-g(0)}{5-0} = f'(c)$$
$$\left|\frac{g(5)-7}{5}\right| = |f'(c)| < 2$$
$$|g(5)-7| < 10$$
$$-10 < g(5) - 7 < 10$$
$$-3 < g(5) < 17.$$

(c) Again, we got the same answer both ways (if we got the absolute values right).

(d) This time, no. Because we bounded the *absolute value* of our derivative, we couldn't go too fast in either direction.