Math 1231: Single-Variable Calculus I
George Washington University Fall 2026
Recitation 9

Jay Daigle

October 21, 2025

Problem 1. Let $f(x) = 2x^3 + 3x^2 - 36x$.

(a) Find the critical points of f.

(b) Which of these points can you classify using the second derivative test?

(c) Classify all the critical points using the first derivative test.

Solution:

(a) $f'(x) = 6x^2 + 6x - 36 = 6(x^2 + x - 6) = 6(x+3)(x-2)$, so the critical points are -3 and 2.

(b) $f''(x) = 12x + 6$. We see $f''(-3) = -30 < 0$ so f has a local maximum of $f(-3) = 81$ at -3. And $f''(2) = 30 > 0$ so f has a local minimum of $f(2) = -44$ at 2.

(c) We can make a chart

	$6(x+3)$	$x-2$	$f'(x)$
$x < -3$	$-$	$-$	$+$
$-3 < x < 2$	$+$	$-$	$-$
$-3 < x < 2$	$+$	$+$	$+-$

So f is increasing on $(-\infty, -3)$ and $(2, +\infty)$ and is decreasing on $(-3, 2)$. So again we see that f has a local max of $f(-3) = 81$ at -3 and a local min of $f(2) = -44$ at 2.

Problem 2. Let $g(x) = \tan(x) - 2x$.

(a) Find the critical points of g.

(b) Which of these critical points can you classify using the second derivative test?

(c) Classify that critical points of g.

Solution:

(a) $g'(x) = \sec^2(x) - 2$. This is undefined when $\sec(x) = 0$, and thus when $x = \pi/2 + n\pi$.

This is zero when $\sec^2(x) = 2$, thus when $\sec(x) = \pm\sqrt{2}$. We check the unit circle: $\sec(\pi/4) = \sqrt{2}$ and $\sec(3\pi/4) = -\sqrt{2}$. Similarly, $\sec(-\pi/4) = \sqrt{2}$ and $\sec(-3\pi/4) = -\sqrt{2}$.

The general set of solutions here is $x = n\pi + \pi/4$ or $x = n\pi + 3\pi/4$. On $(-\pi/2, \pi/2)$, this is just $\pm\pi/4$.

(b) The points where g' is undefined can never be classified with the second derivative test.

For the others, we have $g''(x) = 2\sec^2(x)\tan(x)$. At all those critical points we know $\sec^2(x) = 2$ so this is just $4\tan(x)$. Then

$$g''(\pi/3) = 8\tan(\pi/4) = 4$$
$$g''(-\pi/3) = 8\tan(-\pi/4) = -4$$

and thus we have a relative maximum at $-\pi/4$ and a relative minimum at $\pi/4$.

(Adding a multiple of π to the input doesn't change the value of $\tan(x)$ so we have a relative max at $n\pi - \pi/4$ and a relative min at $n\pi + \pi/4$).

(c) We don't have factors for g' so we can't make our factor chart. But we can check points on the number line.

$$g'(0) = 1 - 2 = -1$$
$$g'(\pi/6) = \frac{4}{3} - 2 = -2/3 \qquad g'(-\pi/6) = \frac{4}{3} - 2 = -2/3$$
$$g'(\pi/3)4 - 2 = 2 \qquad g'(-\pi/3) = 4 - 2 = 2$$
$$g'(2\pi/3) = 4 - 2 = 2 \qquad g'(-2\pi/3) = 4 - 2 = 2$$
$$g'(\pi)1 - 2 = -1 \qquad g'(-\pi) = 1 - 2 = -1$$

And we get a number line something like this:

	$(-5\pi/4, -3\pi/4)$	$(-3\pi/4, -\pi/2)$	$(-\pi/2, -\pi/4)$	$(-\pi/4, \pi/4)$	$(\pi/4, \pi/2)$	$(\pi/2, 3\pi/4)$	$(3\pi/4, 5\pi/4)$
x	$-\pi$	$-2\pi/3$	$-\pi/3$	0	$\pi/3$	$2\pi/3$	π
$g'(x)$	-1	2	2	-1	2	2	-1

This tells us that we have a minimum at $-3\pi/4$ and $\pi/4$ and a maximum at $-\pi/4$ and $3\pi/4$. (And thus, generally, we have minima at $n\pi + \pi/4$ and maxima at $n\pi - \pi/4$). At $\pm\pi/2$ the derivative "changes" from positive to positive, but also the function isn't defined there at all so it can't possibly be a maximum or minimum.

Problem 3 (Bonus)**.** We talked about using the combination of the Intermediate Value Theorem and Rolle's Theorem to figure out exactly how many solutions an equation has.

Let $h(x) = x^5 + x^3 + x - 1$. This is a function which has no roots you can actually write down in a useful way; it's a theorem that you can't give a nice algebraic description of the solutions to $x^5 + x^3 - 1 = 0$. But we can say some things about them.

(a) First, we want to get an idea of how many solutions we expect. Try plugging some small, easy numbers into this function, until you think you understand the function. How many solutions should it have?

(b) Now we want to prove we have to have a solution. Write up an argument in terms of the Intermediate Value Theorem to prove that $x^5 + x^3 - 1 = 0$ has a solution.

(c) Now we'll prove that there can't be any more. Check that this function satisfies the conditions of Rolle's Theorem. How does that prove we can't have two solutions?

(d) Can you explain informally why the equation can only have one solution, in terms of rates of change?

(e) Bonus: could you make the same argument about $h_2(x) = x^5 + x^3 - 1$? Why or why not? What would change?

Solution:

(a)

$$h(-2) = -43 \qquad h(-1) = -4 \qquad h(0) = -1$$
$$h(1) = 2 \qquad h(2) = 41 \qquad h(3) = 272$$

It looks like this function should have a root between $x = 0$ and $x = 1$, but probably not any others.

(b) We know the function h is continuous on the closed interval $[0, 1]$, and we know that $h(0) = -1$ and $h(1) = 2$. Since $-1 < 0 < 2$, then by the Intermediate Value Theorem, there is a c in $(0, 1)$ with $h(c) = 0$.

(c) We know that h is continuous and differentiable everywhere, which lets us use Rolle's Theorem.

Suppose that h has two roots, $a < b$, so that $h(a) = h(b) = 0$. Then h is continuous and differentiable on $[a, b]$ and $h(a) = h(b)$, so by Rolle's Theorem, there is a c in (a, b) so that $h'(c) = 0$.

But $h'(x) = 5x^4 + 3x^2 + 1 \geq 1$, so we can't have a c with $h'(c) = 0$. So we couldn't have two distinct roots for h.

(d) Informally we can say that $h'(x)$ is always positive, so h must always be increasing. And since the value is always increasing, we can't output zero twice: once we've output zero, every further output must be larger.

(e) Not quite! We could make exactly the same argument about the Intermediate Value Theorem: $h_2(0) = -1$ and $h_2(1) = 1$ so by the intermediate value theorem, $h_2(x) = 0$ must have a solution. But $h_2'(x) = 5x^4 + 3x^2$ can in fact be zero, so the Rolle's Theorem argument doesn't actually work.

(This function does have only one root, but we would need slightly better tools to prove this.)

Problem 4. Let $g(x) = x \tan(x)$. We want to sketch a graph of g.

(a) What is the domain of g?

(b) For simplicity, let's just look at $[-\pi/2, \pi/2]$. What can you say about any asymptotes it has?

(c) Does this function have any roots you can find?

(d) $g'(x) = \frac{\sin(x)\cos(x)+x}{\cos^2(x)}$. What are the critical points?

(Hint: when is $\sin(x)\cos(x)$ positive and when is it negative?)

(e) What are the critical values?

(f) Where is g increasing and decreasing? Does it have maxima or minima?

(g) $g''(x) = 2\sec^2(x)(1 + x\tan(x))$. Where are the potential points of inflection, and what are their values? Where is h concave up and down?

(h) Sketch the graph.

Solution:

(a) The domain of g is real numbers except $n\pi + \pi/2$.

(b) In $[-\pi/2, \pi/2]$ we can't include the endpoints so we now have $(-\pi/2, \pi/2)$. $\lim_{x\to -\pi/2^+} g(x) = +\infty$ and $\lim_{x\to \pi/2^-} g(x) = +\infty$, which gives us asymptotes.

(c) The function is 0 when $x = 0$ (and when $x = n\pi$ if we look farther out).

(d) $g'(x) = \tan(x) + x\sec^2(x) = \frac{\sin(x)\cos(x)+x}{\cos^2(x)}$. It's not hard to see that when $-\pi/2 < x < 0$ then $g'(x) < 0$, and when $0 < x < \pi/2$ then $g'(x) > 0$, and $g'(0) = 0$. So the only critical point is at 0.

(e) $g(0) = 0$.

(f) And we saw that g is decreasing on $(-\pi/2, 0)$ and increasing on $(0, \pi/2)$. Thus g has a local minimum at 0. $g(0) = 0$.

(g) $x\tan x \geq 0$ on $(-\pi/2, \pi/2)$, so $g''(x) \geq 0$ on $(-\pi/2, \pi/2)$, so the function is concave up everywhere.

(h)

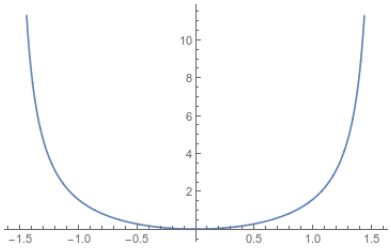


Figure 0.1: The graph of $g(x) = x\tan(x)$

Problem 5. Let $h(x) = \frac{x+2}{x-1}$. We want to sketch a graph of h.

(a) What is the domain of h? What can you say about any asymptotes it has?

(b) Does this function have any roots? Where?

(c) What happens as x approaches $+\infty$? $-\infty$?

(d) $h'(x) = -3(x-1)^{-2}$. What are the critical points and values?

(e) Where is h increasing and decreasing? Does it have maxima or minima?

(f) $h''(x) = 6(x-1)^{-3}$. Where are the potential points of inflection? Where is h concave up and down?

(g) Sketch the graph.

Solution:

(a) The domain of h is all real numbers except 1. We see that $\lim_{x\to 1^-} h(x) = -\infty$ and $\lim_{x\to 1^+} h(x) = +\infty$.

(b) The function has a root at $x = -2$.

(c) We have $\lim_{x\to+\infty} h(x) = \lim_{x\to-\infty} h(x) = 1$. (We can use L'Hôpital's rule or divide the top and bottom by x).

(d) We have $h'(x) = \frac{(x-1)-(x+2)}{(x-1)^2} = -3(x-1)^{-2}$. This has no roots and fails to exist when $x = 1$. Thus there are no "real" critical points.

(e) We make a chart for increase and decrease:

	-3	$(x-1)^{-2}$	$h'(x)$
$x < 1$	$-$	$+$	$-$
$1 < x$	$-$	$+$	$-$

Thus h is decreasing everywhere. It has no local maxima or minima.

(f) $h''(x) = 6(x-1)^{-3}$ is positive when $x > 1$ and negative when $x < 1$, so it is concave down on the left, and concave up on the right.

Problem 6 (Bonus)**.** Let $g(x) = x^5 - 4x^3 + 4x + 7$. We want to sketch a graph of g.

(a) What is the domain of g? What can you say about any asymptotes it has?

(b) Does this function have any roots you can find?

(c) What happens as x approaches $+\infty$? $-\infty$?

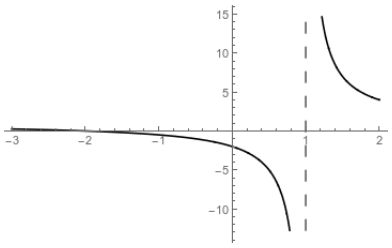


Figure 0.2: The graph of $h(x) = \frac{x+2}{x-1}$

(d) $g'(x) = 5x^4 - 12x^2 + 4$. What are the critical points?

(Hint: if we set $u = x^2$ this becomes a quadratic, and we can factor it.)

(e) What are the critical values?

(Hint: $g(x) = 7 + x(x^4 - 4x^2 + 4) = 7 + x(u^2 - 4u + 4)$.)

(f) Where is g increasing and decreasing? Does it have maxima or minima?

(g) $g''(x) = 20x^3 - 24x$. Where are the potential points of inflection, and what are their values? Where is h concave up and down?

(h) Sketch the graph.

Solution:

(a) The domain of g is all reals. There are no asymptotes.

(b) This function has one real root, but good luck finding it without a computer.

(c) $\lim_{x\to+\infty} g(x) = +\infty$ and $\lim_{x\to-\infty} g(x) = -\infty$.

(d) The derivative is defined everywhere. We can work out that if $u = x^2$ we have

$$4x^5 - 12x^2 + 4 = 5u^2 - 12u + 4 = (5u - 2)(u - 2) = (5x^2 - 2)(x^2 - 2)$$

so the derivative has four roots: where $x^2 = 2$ and where $x^2 = 2/5$. Thus the critical points are $x = \pm\sqrt{2}$ and $x = \pm\sqrt{2/5}$.

(e)

$$\begin{aligned}
g(-\sqrt{2}) &= 7 - \sqrt{2}(4 - 8 + 4) = 7 \\
g(\sqrt{2}) &= 7 + \sqrt{2}(4 - 8 + 4) = 7 \\
g(-\sqrt{2/5}) &= 7 - \sqrt{2/5}\left(\frac{4}{25} - \frac{8}{5} + 4\right) = 7 - \sqrt{2/5} \cdot \frac{64}{25} \\
g(\sqrt{2/5}) &= 7 + \sqrt{2/5}\left(\frac{4}{25} - \frac{8}{5} + 4\right) = 7 + \sqrt{2/5} \cdot \frac{64}{25}.
\end{aligned}$$

(f) We can make a chart:

	$5x^2 - 2$	$x^2 - 2$	g'
$x < -\sqrt{2}$	$+$	$+$	$+$
$-\sqrt{2} < x < -\sqrt{2/5}$	$+$	$-$	$-$
$-\sqrt{2/5} < x < \sqrt{2/5}$	$-$	$-$	$+$
$\sqrt{2/5} < x < \sqrt{2}$	$+$	$-$	$-$
$\sqrt{2} < x$	$+$	$+$	$+$

So f is increasing on $(-\infty, -\sqrt{2}) \cup (-\sqrt{2/5}, \sqrt{2/5}) \cup (\sqrt{2/5}, +\infty)$ and is decreasing on $(-\sqrt{2}, -\sqrt{2/5}) \cup (\sqrt{2/5}, \sqrt{2})$.

g has local maxima at $(-\sqrt{2}, 7)$ and at $\left(\sqrt{2/5}, 7 + \sqrt{2/5} \cdot \frac{64}{25}\right)$. It has local minima at $\left(-\sqrt{2/5}, 7 - \sqrt{2/5} \cdot \frac{64}{25}\right)$ and $(\sqrt{2}, 7)$.

(g) $g''(x) = 4x(5x^2 - 6)$ so the possible points of inflection are 0 and $\pm\sqrt{6/5}$. We compute

$$\begin{aligned}
g(-\sqrt{6/5}) &= 7 - \sqrt{6/5}\left(\frac{36}{25} - \frac{24}{5} + 4\right) = 7 - \sqrt{6/5} \cdot \frac{16}{25} \\
g(0) &= 7 \\
g(\sqrt{6/5}) &= 7 + \sqrt{6/5}\left(\frac{36}{25} - \frac{24}{5} + 4\right) = 7 + \sqrt{6/5} \cdot \frac{16}{25}
\end{aligned}$$

To find concavity we can make another chart:

	$4x$	$5x^2 - 6$	g'
$x < -\sqrt{5/6}$	$-$	$+$	$-$
$-\sqrt{6/5} < x < 0$	$-$	$-$	$+$
$0 < x < \sqrt{6/5}$	$+$	$-$	$-$
$\sqrt{6/5} < x$	$+$	$+$	$+$

So g is concave upwards on $(-\sqrt{6/5}, 0) \cup (\sqrt{6/5}, +\infty)$, and concave downwards on $(-\infty, -\sqrt{6/5}) \cup (0, \sqrt{6/5})$. (So all the potential points of inflection are in fact points of inflection.)

(h)

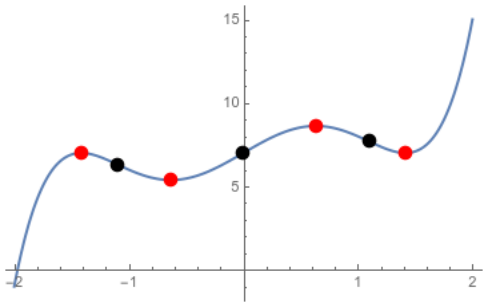


Figure 0.3: The graph of $g(x) = x^5 - 4x^3 + 4x + 7$. Critical points in red, and points of inflection in black.