Hamilton's Method

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Notation

Definition

- An apportionment method is a function whose input is a census $h, n, p_1, \ldots, p_n$, and whose output is a collection of positive integers $a_1, a_2, \ldots, a_n$ that add up to h.
- We define a state's standard quota to be the number $q_k = h \cdot \frac{p_k}{p}$.
- We define a census's standard divisor to be the number $s = \frac{p}{h}$.
- The lower quota for state k is $\lfloor q_k \rfloor$, the standard quota rounded down.
- The upper quota for state k is $\lceil q_k \rceil$ the standard quota rounded up.

Hamilton's Method

Definition (Hamilton's method)

- As a provisional apportionment, assign each state its lower quota $\lfloor q_k \rfloor$.
- Then assign the seats that remain to the states in decreasing order of the size of the fractional parts of their standard quotas, allocating at most one per state.

Hamilton's Method: the Alabama Paradox

k	p_k	$h = 10$	$h = 11$
1	1,450,000	2	1
2	3,400,000	3	4
3	5,150,000	5	6

Definition

When adding a house seat would cause a state to lose a representative, we call that the Alabama paradox.

- Could have happened to Alabama in 1880
- Seems unfair!

House Monotonicity

Definition

- An apportionment method is called house monotone if an increase in h, while all other parameters remain the same, can never cause any seat allocation a_k to decrease.
- Hamilton's method is not house monotone.

Hamilton's Method: the Population Paradox

k	p_k	q_k	$\lfloor q_k \rfloor$	$\{q_k\}$	Hamilton Apportionment
1	1,450,000	1.45	1	0.45	2
2	3,400,000	3.40	3	0.40	3
3	5,150,000	5.15	5	0.15	5

Example

- New census: populations 1,470,000; 3,380,000; 4,650,000.
- Total population $p = 9{,}500{,}000$
- New $s = 950{,}000$.

Hamilton's Method: the Population Paradox

k	p_k	q_k	$\lfloor q_k \rfloor$	$\{q_k\}$	Hamilton Apportionment
1	1,450,000	1.45	1	0.45	2
2	3,400,000	3.40	3	0.40	3
3	5,150,000	5.15	5	0.15	5

k	p_k	q_k	$\lfloor q_k \rfloor$	$\{q_k\}$	Hamilton Apportionment
1	1,470,000	1.55	1	0.55	1
2	3,380,000	3.56	3	0.56	4
3	4,650,000	4.89	4	0.89	5

Hamilton's Method: the Population Paradox

k	Census 1 p_k	Census 1 a_k	Census 2 p_k	Census 2 a_k
1	1,450,000	2	1,470,000	1
2	3,400,000	3	3,380,000	4
3	5,150,000	5	4,650,000	5

Definition

- State 1 gained population; states 2 and 3 lost population
- But state 1 lost representation
- Seems *really* unfair!

Population Monotonicity

Definition

- A method is called *population monotone* if a state can never lose a seat when its population increases while no other state's population increases.
- In algebraic terms, whenever $a_i' < a_i$ and $a_j' > a_j$, it must be the case either that $p_i' < p_i$ or $p_j' > p_j$.

- Hamilton's method is not population monotone.

Hamilton's Method: the Oklahoma Paradox

k	p_k	q_k	$\lfloor q_k \rfloor$	$\{q_k\}$	Hamilton Apportionment
1	1,450,000	1.45	1	0.45	2
2	3,400,000	3.40	3	0.40	3
3	5,150,000	5.15	5	0.15	5

Example

- Add a new state, $p_4 = 2{,}600{,}000$.
- Add 3 seats to the house, so $h = 13$
- New $p = 12{,}600{,}000$

Hamilton's Method: the Oklahoma Paradox

k	p_k	q_k	$\lfloor q_k \rfloor$	$\{q_k\}$	Hamilton Apportionment
1	1,450,000	1.50	1	0.50	1
2	3,400,000	3.51	3	0.51	4
3	5,150,000	5.31	5	0.31	5
4	2,600,000	2.68	2	0.68	3

- $p = 12{,}600{,}000$, $h = 13$, so $s = \frac{12{,}600{,}000}{13} \approx 969{,}231$.
- What happens?

Hamilton's Method: the Oklahoma Paradox

k	Census 1 p_k	Census 1 a_k	Census 2 p_k	Census 2 a_k
1	1,450,000	2	1,450,000	1
2	3,400,000	3	3,400,000	4
3	5,150,000	5	5,150,000	5
4	0	0	2,600,000	3

- Adding a new state moved a representative from State 1 to State 2.
- Nothing about State 1 or State 2 changed
- Should State 1 vote to admit a new state?

Hamilton's Method: the Oklahoma Paradox

Definition

When adding a new state would cause a pre-existing state to lose a representative to a different pre-existing state, we call that the Oklahoma paradox.

- Nearly happened in 1907
- Admitted Oklahoma, added 5 house seats for them.
- Hamilton's method did give Oklahoma 5 seats
- But! In 1900, New York got 38 seats and Maine got 3
- Re-calculating would have given New York 37 and Maine 4
- Never used Hamilton's method again.

Hamilton's method

- Simple, obvious answer to the apportionment problem
- Suggestion dates to 1792
- Officially in use 1850 – 1890; almost used in 1900.
- (Method was mostly ignored in 1860 and 1870.)
- Paradoxes led to vicious Congressional arguments about the value of h
- Permanently abandoned.

Discussion Question

What else could we do?

Jefferson's Method

- Hamilton's method was *not* used at the founding
- Not obvious that size of Congress should be fixed!

US Constitution Article I Section 2

Representatives and direct Taxes shall be apportioned among the several States which may be included within this Union, according to their respective Numbers . . .
The Number of Representatives shall not exceed one for every thirty Thousand, but each State shall have at Least one Representative . . .

- Jefferson wanted to fix the size of a *district* instead.

Proposed Bill of Rights

Article the First

After the first enumeration required by the first article of the Constitution, there shall be one Representative for every thirty thousand, until the number shall amount to one hundred, after which the proportion shall be so regulated by Congress, that there shall be not less than one hundred Representatives, nor less than one Representative for every forty thousand persons, until the number of Representatives shall amount to two hundred; after which the proportion shall be so regulated by Congress, that there shall not be less than two hundred Representatives, nor more than one Representative for every fifty thousand persons.

- One state short of ratification in 1791
- Never ratified
- Would require six thousand Congresspeople today.

Jefferson's Method

- Hamilton: computed standard divisor $s = p/h$.
- s is the ideal size of a Congressional district, given h.
- Jefferson wanted to *start* with a district size d and compute h

- From the Constitution, set $d = 30{,}000$. Call this a modified divisor.
- Compute modified quotas p_k/d
- This isn't a whole number
- Round down, set $a_k = \lfloor p_k/d \rfloor$.

Discussion Question

Why did Jefferson suggest rounding down rather than up?

Jefferson's Method

- Jefferson's original approach is not an "apportionment method" by our definition
- Why not?
- But we can modify it.

Definition (Jefferson's method)

- Choose a modified divisor d
- Compute the modified quotas p_k/d
- Round these down to obtain $a_k = \lfloor p_k/d \rfloor$.
- If $a_1 + a_2 + \cdots + a_n = h$, then we have the Jefferson apportionment.
- Otherwise, choose a new d and try again.

Jefferson's Method

Definition (Jefferson's method)

- Choose a modified divisor d
- Compute modified quotas p_k/d
- Round these down to obtain $a_k = \lfloor p_k/d \rfloor$.
- If $a_1 + a_2 + \cdots + a_n = h$, then we have the Jefferson apportionment. Otherwise, try again.

Questions about Jefferson's Method

1. Is there always a d that will work? Can we find it?
2. Is there ever *more than one* d that will work?
3. If we pick two different ds that both give the same total number of seats, will they give the same apportionment?

Jefferson's Method

Is there always a d that works?

- Yes!
- As long as we ignore exact ties.

How can we find d?

- If we allocate too many seats: d was too small
- If we don't allocate enough seats: d was too big
- Can home in on a d that works quickly.
- We can systematize this more, and will soon.

Jefferson's Method

Example

Find the Jefferson apportionment with $n = 3$, $h = 10$, and state populations $p_1 = 1{,}500{,}000$, $p_2 = 3{,}200{,}000$, $p_3 = 5{,}300{,}000$.

		$s =$1,000,000		$d = 900{,}000$		$d = 800{,}000$		$d = 850{,}000$	
k	p_k	q_k	$\lfloor q_k \rfloor$	q	$\lfloor q \rfloor$	q	$\lfloor q \rfloor$	q	$\lfloor q \rfloor$
1	1,500,000	1.50	1	$1.\overline{66}$	1	1.875	1	1.76	1
2	3,200,000	3.20	3	$3.\overline{55}$	3	4	4	3.67	3
3	5,300,000	5.30	5	$5.\overline{88}$	5	6.625	6	6.24	6
	10,000,000		9		9		11		10

Jefferson's Method

Can more than one d work?

- Absolutely!

		$d = 850{,}000$		$d = 860{,}000$	
k	p_k	q	$\lfloor q \rfloor$	q	$\lfloor q \rfloor$
1	1,500,000	1.76	1	1.74	1
2	3,200,000	3.67	3	3.72	3
3	5,300,000	6.24	6	6.16	6
	10,000,000		10		10

Does it matter which d we pick?

- Nope!

Jefferson's Method

Proposition

Suppose h, n, and $p_1, \ldots, p_n$ are given as inputs to our apportionment function. If d and d' are two different divisors, yielding Jefferson apportionments $a_1, \ldots, a_n$ and $a'_1, \ldots, a'_n$ respectively, then $a_k = a'_k$ for each state k.

Proof.

- Next time!

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