Jefferson's Method

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Jefferson's Method

Definition (Jefferson's method)

- Choose a modified divisor d
- Compute the modified quotas p_k/d
- Round these down to obtain $a_k = \lfloor p_k/d \rfloor$.
- If $a_1 + a_2 + \cdots + a_n = h$, then we have the Jefferson apportionment.
- Otherwise, choose a new d and try again.

Jefferson's Method

Example

Find the Jefferson apportionment with $n = 3$, $h = 10$, and state populations $p_1 = 1{,}500$, $p_2 = 3{,}200$, $p_3 = 5{,}300$.

		$s =$1,000		$d = 900$		$d = 800$		$d = 850$	
k	p_k	q_k	$\lfloor q_k \rfloor$	q	$\lfloor q \rfloor$	q	$\lfloor q \rfloor$	q	$\lfloor q \rfloor$
1	1,500	1.50	1	$1.\overline{66}$	1	1.875	1	1.76	1
2	3,200	3.20	3	$3.\overline{55}$	3	4	4	3.67	3
3	5,300	5.30	5	$5.\overline{88}$	5	6.625	6	6.24	6
	10,000		9		9		11		10

Questions about Jefferson's Method

1. Is there always a d that will work?
   - Yes, if we don't have ties
2. Can we find it?
   - Yes!
   - Guess a random d
   - If it gives too many seats, pick a larger number
   - If it doesn't give enough seats, pick a smaller number
3. Is there ever *more than one* d that will work?
   - Yes
4. If we pick two different ds that both give the same total number of seats, will they give the same apportionment?
   - Fortunately, yes.

Jefferson's Method

Proposition

Suppose $h, n,$ *and* $p_1, \ldots, p_n$ *are given as inputs to our apportionment function. If* d *and* d' *are two different divisors, yielding Jefferson apportionments* $a_1, \ldots, a_n$ *and* $a'_1, \ldots, a'_n$ *respectively, then* $a_k = a'_k$ *for each state* k*.*

Proof.

- Suppose "without loss of generality" that $d \leq d'$.
- For every state, $p_k/d \geq p_k/d'$.
- Rounding down doesn't change that, so $a_k \geq a'_k$.
- Both apportionments give h seats, so
  $a_1 + a_2 + \cdots + a_n = a'_1 + a'_2 + \cdots + a'_n$.

Jefferson's Method

Proposition

Suppose $h, n,$ *and* $p_1, \ldots, p_n$ *are given as inputs to our apportionment function. If* d *and* d' *are two different divisors, yielding Jefferson apportionments* $a_1, \ldots, a_n$ *and* $a'_1, \ldots, a'_n$ *respectively, then* $a_k = a'_k$ *for each state* k.

Proof.

- For every state, $a_k \geq a'_k$
- $h = a_1 + a_2 + \cdots + a_n \geq a'_1 + a'_2 + \cdots + a'_n = h$
- No number on the left can be smaller, so none of them can be bigger.
- Therefore, for each k, $a_k = a'_k$.

□

How do we find the right divisor?

Trial and error

- Guess randomly
- Will work eventually probably.

Binary search

- Guess a divisor
- If it's too big, pick a smaller one
- If it's too small, pick a bigger one
- If one is too small and another is too big, try halfway between
- Keep trying halfway points until one works.

- There's a better way
- Understand *why* a divisor will work

Critical Divisors

- Small changes in d rarely matter

		$d = 850$		$d = 860$		$d = 870$		$d = 880$		$d = 890$	
k	p_k	q	$\lfloor q \rfloor$	q	$\lfloor q \rfloor$	q	$\lfloor q \rfloor$	q	$\lfloor q \rfloor$	q	$\lfloor q \rfloor$
1	1,500	1.76	1	1.74	1	1.72	1	1.70	1	1.69	1
2	3,200	3.67	3	3.72	3	3.68	3	3.64	4	3.60	4
3	5,300	6.24	6	6.16	6	6.09	6	6.02	6	5.96	5
	10,000		10		10		10		10		9

- When will it change?

Critical Divisors

- Had $p_3 = 5{,}300$
- If $d = 850$ then $q = \frac{5{,}300}{850} \approx 6.24$.
- 6.24 “rounds down to” 6. What does that mean?
- $6 \leq 6.24 < 7$ that is, $6 \leq \frac{5{,}300}{850} < 7$
- $850 \leq \frac{5{,}300}{6}$ but also $\frac{5{,}300}{7} < 850$.
- When does the rounding change?
  - When $d = \frac{5{,}300}{6} = 883.3$ or $d = \frac{5{,}300}{7} = 757$.

Definition

We call a number of the form $\frac{p_k}{m}$ for a positive integer k a (Jefferson) critical divisor for the state k.

Critical Divisors

Definition

We call a number of the form $\frac{p_k}{m}$ for a positive integer k a (Jefferson) critical divisor for the state k.

- Imagine d is very very large. What happens?
  - Every state gets no representatives
- Decrease d slowly. When will a state first get a representative?
  - When d gets less than the population.
  - That means the state contains a whole district.
- When will a state get its second representative?
  - When the state contains two districts
  - When d is less than *half* the population.
- When will that state get its third representative?

Critical Divisors

Definition

We call a number of the form $\frac{p_k}{m}$ for a positive integer k a (Jefferson) critical divisor for the state k.

- Suppose $p_3 = 5,300$.

m	1	2	3	4	5	6
$\frac{p_k}{m}$	5,300	2,650	1,767	1,325	1,060	883

- If $d > 5,300$, then $a_k = 0$.
- If $5,300 > d > 2,650$, then $a_k = 1$
- If $2,650 > d > 1,767$, then $a_k = 2$
- When will $a_k = 5$?
  - When $1,060 > d > 883$.

Critical Divisors

Computing with Critical Divisors

- If d is very big, give zero seats
- Shrink d. Every time we cross a critical divisor for a state, that state gets one more representative.
- We want to give out h seats in total
- So we give seats to the biggest h critical divisors
- Can pick any d between divisors h and $h+1$.

Critical Divisors

Example

Find the Jefferson apportionment with $n = 3, h = 10$, and state populations $p_1 = 1{,}500$, $p_2 = 3{,}200$, $p_3 = 5{,}300$.

d	State 1	State 2	State 3
1	$\frac{1{,}500}{1} = 1,500$	$\frac{3{,}200}{1} = 3,200$	$\frac{5{,}300}{1} = 5,300$
2	$\frac{1{,}500}{2} = 750$	$\frac{3{,}200}{2} = 1,600$	$\frac{5{,}300}{2} = 2,650$
3	$\frac{1{,}500}{3} = 500$	$\frac{3{,}200}{3} = 1,067$	$\frac{5{,}300}{3} = 1,767$
4	$\frac{1{,}500}{4} = 375$	$\frac{3{,}200}{4} = 800$	$\frac{5{,}300}{4} = 1,325$
5	$\frac{1{,}500}{5} = 300$	$\frac{3{,}200}{5} = 640$	$\frac{5{,}300}{5} = 1,060$
6	$\frac{1{,}500}{6} = 250$	$\frac{3{,}200}{6} = 533$	$\frac{5{,}300}{6} = 883$
7	$\frac{1{,}500}{7} = 215$	$\frac{3{,}200}{7} = 457$	$\frac{5{,}300}{7} = 757$

Critical Divisors

Advantages and Disadvantages

- No trial and error
- Straightforward calculation
- A *lot* of straightforward calculation

Can we speed this up?

- Can calculate s easily
- Good first guess, but d should always be smaller
- Count down from s.

Critical Divisors

Example

If $n = 3, h = 10,\ p_1 = 1{,}500,\ p_2 = 3{,}200,$ and $p_3 = 5{,}300$ then $p = 10{,}000$ and $s = 1{,}000$

d	State 1	State 2	State 3
1	$\frac{1{,}500}{1} = 1,500$	$\frac{3{,}200}{1} = 3,200$	$\frac{5{,}300}{1} = 5,300$
2	$\frac{1{,}500}{2} = 750$	$\frac{3{,}200}{2} = 1,600$	$\frac{5{,}300}{2} = 2,650$
3	$\frac{1{,}500}{3} = 500$	$\frac{3{,}200}{3} = 1,067$	$\frac{5{,}300}{3} = 1,767$
4	$\frac{1{,}500}{4} = 375$	$\frac{3{,}200}{4} = 800$	$\frac{5{,}300}{4} = 1,325$
5	$\frac{1{,}500}{5} = 300$	$\frac{3{,}200}{5} = 640$	$\frac{5{,}300}{5} = 1,060$
6	$\frac{1{,}500}{6} = 250$	$\frac{3{,}200}{6} = 533$	$\frac{5{,}300}{6} = 883$
7	$\frac{1{,}500}{7} = 215,$	$\frac{3{,}200}{7} = 457$	$\frac{5{,}300}{7} = 757$

Critical Divisors

- Even better: don't have to calculate them all
- Start from standard divisor s
- Find round-down apportionments $\lfloor q_k \rfloor$
- Lose a seat when d rises to $\frac{p_k}{\lfloor q_k \rfloor}$
- Gain a seat when d drops to $\frac{p_k}{\lfloor q_k \rfloor + 1}$.
- So calculate $\frac{p_k}{\lfloor q_k \rfloor + 1}$ for each k, see which comes first.

Critical Divisors

Example

If $n = 3, h = 10$, $p_1 = 1{,}500$, $p_2 = 3{,}200$, and $p_3 = 5{,}300$ then $p = 10{,}000$ and $s = 1{,}000$

		$s = 1,000$		Next Divisor	$d = 850$	
k	p_k	q_k	$\lfloor q_k \rfloor$	$\frac{p_k}{\lfloor q_k \rfloor + 1}$	q	a_k
1	1,500	1.50	1	750	1.76	1
2	3,200	3.20	3	800	3.76	3
3	5,300	5.30	5	883	6.24	6
Total	10,000		9			10

Properties of Jefferson's Method

- Favors large states relative to Hamilton's method

Example

Find the Hamilton and Jefferson apportionments when

$n = 2, h = 10$, with $p_1 = 1{,}800$ and $p_2 = 8{,}200$.

k	p_k	q_k	$\lfloor q_k \rfloor$	Ham a_k	CD	$d = 910$	Jef a_k
1	1,800	1.8	1	2	900	1.98	1
2	8,200	8.2	8	8	911	9.02	9

Properties of Jefferson's Method

Example

Find the Hamilton and Jefferson apportionments when $n = 4, h = 10$, with populations below:

k	p_k	q_k	$\lfloor q_k \rfloor$	Ham a_k	CD	$d = 725$	Jeff a_k
1	1,500	1.5	1	2	750	2.07	2
2	1,400	1.4	1	1	700	1.93	1
3	1,300	1.3	1	1	650	1.79	1
4	5,800	5.8	5	6	967	8.00	8
	10,000		8	10			12 ??

- That didn't work
- What went wrong?

Properties of Jefferson's Method

Example

Find the Hamilton and Jefferson apportionments when

$n = 4$, $h = 10$, with populations below:

k	p_k	q_k	$\lfloor q_k \rfloor$	Ham a_k	CD		CD	
1	1,500	1.5	1	2	750	1	750	1
2	1,400	1.4	1	1	700	1	700	1
3	1,300	1.3	1	1	650	1	700	1
4	5,800	5.8	5	6	967	6	829	7
	10,000		8	10		9		10

Properties of Jefferson's Method

Example

Find the Hamilton and Jefferson apportionments when

$n = 4$, $h = 10$, with populations below:

k	p_k	q_k	$\lfloor q_k \rfloor$	Ham a_k	CD	$d = 800$	Jeff a_k
1	1,500	1.5	1	2	750	1.88	1
2	1,400	1.4	1	1	700	1.75	1
3	1,300	1.3	1	1	650	1.62	1
4	5,800	5.8	5	6	967	7.25	7
	10,000		8	10			10

Properties of Jefferson's Method

Important Computational Note

- Hamilton's method rounds states up, one by one
- Jefferson's method doesn't do that
- Some states can get bumped up more than once
- To use critical divisors:
  - Compute critical divisors
  - Give seat to state with largest critical divisor
  - *Compute critical divisors again*
  - Allocate seat again
  - Repeat until all excess seats are allocated.

- But also we get a weird result here, right?

Quota Rules

Definition

- We say it's a quota violation if an apportionment method gives a state more representatives than its upper quota, or less than its lower quota.
- An apportionment method satisfies the quota rule if it assigns every state either its lower quota or its upper quota.

Proposition

Jefferson's method violates the quota rule.

- Sometimes useful to split this into two ideas.

Quota Rules

Definition

- An apportionment method satisfies the upper quota rule if it never assigns a state more than its upper quota.
- A violation of this rules is an upper quota violation.

Definition

- An apportionment method satisfies the lower quota rule if it never assigns a state less than its lower quota.
- A violation of this rules is an lower quota violation.

Proposition

Jefferson's method violates the upper quota rule.

Quota Rules

Proposition

Jefferson's method satisfies the lower quota rule.

Proof.

- Jefferson apportionment with $d = s$ gives every state $\lfloor q_k \rfloor$
- That won't allocate enough seats
- Need to pick a smaller d
- That will never give any state *fewer* seats than s would
- Every state gets at least its lower quota.

□

Jefferson and Monotonicity

Proposition

Jefferson's method is house monotone.

Proof.

Next time! □