Divisor Methods

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Jefferson's Method

Definition (Jefferson's method)

- Choose a modified divisor d
- Compute the modified quotas p_k/d
- Round these down to obtain $a_k = \lfloor p_k/d \rfloor$.
- If $a_1 + a_2 + \cdots + a_n = h$, then we have the Jefferson apportionment.
- Otherwise, choose a new d and try again.

Jefferson and Monotonicity

Proposition

Jefferson's method is house monotone.

Proof 1 with Modified Divisors.

- Increasing h will require a smaller divisor d
- Decreasing d gives a higher p_k/d to each state
- Rounding down a larger number will never give a smaller number.
- (It might not give a bigger number either but that's fine.)
- So reducing d will never cause a state to lose a seat
- Therefore increasing h will never cause a state to lose a seat.

□

Jefferson and Monotonicity

Proposition

Jefferson's method is house monotone.

Proof 2 with Critical Divisors.

- List all critical divisors in decreasing order
- We allocate h seats by choosing the first h divisors in the list
- If we increase h, we get more divisors
- But this will include all the divisors we got the first time
- We allocate all the original seats, and then allocate more
- Therefore no state will lose a seat when we increase h.

□

Adams's Method

- Do we have to round down?

Definition (Adams's method)

- Choose a modified divisor d
- Compute the modified quotas p_k/d
- Round these *up* to obtain $a_k = \lceil p_k/d \rceil$.
- If $a_1 + a_2 + \cdots + a_n = h$, then we have the Adams apportionment.
- Otherwise, choose a new d and try again.

- Jefferson's method asks how many districts fit *inside* a state
- Adams's method asks how many districts you need to *cover* a state.

Adams's Method

Example

Find the Jefferson and Adams apportionments when $n = 2, h = 10$, with $p_1 = 1{,}800$ and $p_2 = 8{,}200$.

- We've seen Jefferson already
- Can use guess-and-check, binary search, or critical divisors

k	p_k	q_k	$\lfloor q_k \rfloor$	Ham	CD	$d = 910$	Jef a_k
1	1,800	1.8	1	2	900	1.98	1
2	8,200	8.2	8	8	911	9.02	9

Adams's Method

Example

Find the Jefferson and Adams apportionments when $n = 2$, $h = 10$, with $p_1 = 1{,}800$ and $p_2 = 8{,}200$.

		$s = 1{,}000$			$d = 910$			$d = 1{,}100$		
k	p_k	q_k	$\lfloor q_k \rfloor$	$\lceil q_k \rceil$	q	$\lfloor q \rfloor$	$\lceil q_k \rceil$	q	$\lfloor q \rfloor$	$\lceil q_k \rceil$
1	1,800	1.8	1	2	1.98	1	2	1.63	1	2
2	8,200	8.2	8	9	9.02	9	10	7.45	7	8
			9	11		10	12		8	10

Adams's Method

- Can also use a critical divisors approach
- The interesting divisors are the same
  - $q = 5.2$ and $q = 5.8$ give the same result
  - but $q = 6.1$ is different
  - Interesting divisors happen at p_k/m, integer fractions of the population.
- But something changes.
- What happens when d is very big?
  - Every state still gets a representative
  - Then gets one *more* every time we cross a critical divisor.

Adams's Method

Jefferson Divisors		
d	State 1	State 2
1	$\frac{1{,}800}{1} = 1,800$	$\frac{8{,}200}{1} = 8,200$
2	$\frac{1{,}800}{2} = 900$	$\frac{8{,}200}{2} = 4,100$
3	$\frac{1{,}800}{3} = 600$	$\frac{8{,}200}{3} = 2,733$
4	$\frac{1{,}800}{4} = 450$	$\frac{8{,}200}{4} = 2,050$
5	$\frac{1{,}800}{5} = 360$	$\frac{8{,}200}{5} = 1,640$
6	$\frac{1{,}800}{6} = 300$	$\frac{8{,}200}{6} = 1,367$
7	$\frac{1{,}800}{7} = 257$	$\frac{8{,}200}{7} = 1,171$
8	$\frac{1{,}800}{8} = 225$	$\frac{8{,}200}{8} = 1,025$
9	$\frac{1{,}800}{9} = 200$	$\frac{8{,}200}{9} = 911$

Adams Divisors		
d	State 1	State 2
0	∞	∞
1	$\frac{1{,}800}{1} = 1,800$	$\frac{8{,}200}{1} = 8,200$
2	$\frac{1{,}800}{2} = 900$	$\frac{8{,}200}{2} = 4,100$
3	$\frac{1{,}800}{3} = 600$	$\frac{8{,}200}{3} = 2,733$
4	$\frac{1{,}800}{4} = 450$	$\frac{8{,}200}{4} = 2,050$
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7	$\frac{1{,}800}{7} = 257$	$\frac{8{,}200}{7} = 1,171$
8	$\frac{1{,}800}{8} = 225$	$\frac{8{,}200}{8} = 1,025$

Adams's Method

- $d = s$ will be too *low*
- Need to slowly *increase* it to find the right divisor

k	p_k	q_k	$\lfloor q_k \rfloor$	$\frac{p_k}{\lfloor q_k \rfloor + 1}$	$\frac{p_k}{\lfloor q_k \rfloor}$	$d = 1{,}100$	Adams a_k
1	1,800	1.8	1	900	1,800	1.63	2
2	8,200	8.2	8	911	1,025	7.45	8

Adams's Method

Example

Find the Hamilton, Jefferson and Adams apportionments when $n = 2$, $h = 10$, with $p_1 = 1{,}200$ and $p_2 = 8{,}800$.

k	p_k	q_k	$\lfloor q_k \rfloor$	Ham	$\frac{p_k}{\lfloor q_k \rfloor + 1}$	$d = 900$	Jef
1	1,200	1.2	1	1	600	1.33	1
2	8,800	8.8	8	9	978	9.78	9

k	p_k	q_k	$\lfloor q_k \rfloor$	$\lceil q_k \rceil$	$\frac{p_k}{\lfloor q_k \rfloor}$	$d = 1{,}150$	Adams
1	1,200	1.2	1	2	1,200	1.04	2
2	8,800	8.8	8	9	1,100	7.65	8

Adams's Method

Example

Find the Adams apportionments when $n = 4$, $h = 10$, with populations below:

k	p_k	q_k	$\lceil q_k \rceil$	$\frac{p_k}{\lceil q_k \rceil - 1}$	Temp a_k	$\frac{p_k}{a_k - 1}$	
1	1,500	1.5	2	1,500	2	1,500	2
2	1,400	1.4	2	1,400	2	1,400	2
3	1,300	1.3	2	1,300	2	1,300	1
4	5,800	5.8	6	1,160	5	1,450	5
	10,000		12		11		10

- Should pick $d = 1{,}350$.

Adams's Method

Example

Find the Adams apportionments when $n = 4$, $h = 10$, with populations below:

k	p_k	q_k	$\lceil q_k \rceil$	Ham a_k	$d = 1{,}350$	Adams a_k
1	1,500	1.5	2	2	1.11	2
2	1,400	1.4	2	1	1.04	2
3	1,300	1.3	2	1	0.96	1
4	5,800	5.8	6	6	4.29	5
	10,000		12	10		10

Adams's Method

- Adams's method favors small states
- Adams's method automatically guarantees each state at least one seat.

Proposition

Adams's method violates the lower quota rule.

Proof.

- Proof by example
- Not hard to find a case where a large state gets less than its lower quota.

□

Splitting the difference: Webster's Method

- Jefferson's method rounds down, favors large states
- Adams's method rounds up, favors small states
- Split the difference and round "normally"?
  - We can call this "normal" rounding or "grade-school" rounding
  - A fancier name is arithmetic rounding. We'll talk more about this next time.

Splitting the difference: Webster's Method

Definition (Webster's method)

- Choose a modified divisor d
- Compute the modified quotas p_k/d
- Round these to the nearest whole number to obtain a_k.
- If $a_1 + a_2 + \cdots + a_n = h$, then we have the Webster apportionment.
- Otherwise, choose a new d and try again.

Webster's Method

- In Jefferson's method, s is always too big.
- In Adams's method, s is always too small.
- In Webster's method, it could be too big, or too small, or just right.
- Think about critical divisors bigger *and* smaller.

Which divisors are critical?

- Whole numbers aren't the important ones
- Round down at 3.49 but up at 3.5
- Want to look at $\frac{p}{m+1/2}$.

Webster's Method

Example

Find the Webster apportionments when $n = 4, h = 10$, with populations below:

k	p_k	q_k	$\lfloor q_k \rfloor$	Ham a_k	Arithmetic rounding	Webster a_k
1	1,500	1.5	1	2	2	2
2	1,400	1.4	1	1	1	1
3	1,300	1.3	1	1	1	1
4	5,800	5.8	5	6	6	6
	10,000		8	10		10

- Take $d = 1{,}000$.

Rounding

Discussion Question

- What number is halfway between 2 and 4?
- What number is halfway between 1 and 100?
- What number is halfway between 1 and $\frac{1}{100}$?
- What does "halfway" mean?