# Divisor Methods

Jay Daigle
jaydaigle@gwu.edu
https://jaydaigle.net/politics

The George Washington University

Tuesday, March 3, 2026

# Rounding

## Discussion Question

- What number is halfway between 2 and 4?
- What number is halfway between 1 and 100?
- What number is halfway between $\frac{1}{2}$ and $\frac{1}{4}$?
- What does "halfway" mean?

# Jefferson's Method

**Definition (Jefferson's method)**

- Choose a modified divisor $d$
- Compute the modified quotas $p_k/d$
- Round these down to obtain $a_k = \lfloor p_k/d \rfloor$.
- If $a_1 + a_2 + \cdots + a_n = h$, then we have the Jefferson apportionment.
- Otherwise, choose a new $d$ and try again.

# Adams's Method

**Definition (Adams's method)**

- Choose a modified divisor $d$
- Compute the modified quotas $p_k/d$
- Round these *up* to obtain $a_k = \lceil p_k/d \rceil$.
- If $a_1 + a_2 + \cdots + a_n = h$, then we have the Adams apportionment.
- Otherwise, choose a new $d$ and try again.

# Splitting the difference: Webster's Method

- Jefferson's method rounds down, favors large states
- Adams's method rounds up, favors small states
- Split the difference and round normally?

**Definition (Webster's method)**

- Choose a modified divisor $d$
- Compute the modified quotas $p_k/d$
- Round these to the nearest whole number to obtain $a_k$.
- We've been calling this "grade-school rounding"; a fancier name is arithmetic rounding.
- If $a_1 + a_2 + \cdots + a_n = h$, then we have the Webster apportionment.
- Otherwise, choose a new $d$ and try again.

# Webster's Method

- In Jefferson's method, $s$ is always too big.
- In Adams's method, $s$ is always too small.
- In Webster's method, it could be too big, too small, or just right.
- Think about bigger *and* smaller critical divisors.

## Which divisors are critical?

- Whole numbers aren't the important ones
- Round down at 3.49 but up at 3.5
- Need to look at $\frac{p_k}{2.5}$ and $\frac{p_k}{3.5}$, etc.
- Want to look at $\frac{p_k}{m+1/2}$.

# Webster's Method

## Example

Apportion $h = 10$ seats to $n = 3$ states with populations 3,300, 5,100, and 1,600, using the methods of Hamilton and Jefferson.

- Next Jefferson critical divisor: $\frac{p_k}{\lfloor q_k \rfloor + 1} = \frac{p_k}{a_k + 1}$.

| | | $s = 1{,}000$ | | | Jefferson $d = 840$ | | |
|---|---|---|---|---|---|---|---|
| $k$ | $p_k$ | $q_K$ | $\lfloor q_k \rfloor$ | Ham | CD $\frac{p_k}{a_k+1}$ | Jef $q$ | Jef $a_k$ |
| 1 | 3,300 | 3.3 | 3 | 3 | 825 | 3.93 | 3 |
| 2 | 5,100 | 5.1 | 5 | 5 | 850 | 6.07 | 6 |
| 3 | 1,600 | 1.6 | 1 | 2 | 800 | 1.90 | 1 |

# Webster's Method

**Example**

Apportion $h = 10$ seats to $n = 3$ states with populations 3,300, 5,100, and 1,600, using the method of Adams.

- *Previous* Adams critical divisor: $\frac{p_k}{\lceil q_k \rceil - 1} = \frac{p_k}{a_k - 1}$.

| | | $s = 1{,}000$ | | | Adams $d = 1{,}150$ | | |
|---|---|---|---|---|---|---|---|
| $k$ | $p_k$ | $q_K$ | $\lfloor q_k \rfloor$ | Ham | CD $\frac{p_k}{a_k - 1}$ | Adams $q$ | Adams $a_k$ |
| 1 | 3,300 | 3.3 | 3 | 3 | 1100 | 2.86 | 3 |
| 2 | 5,100 | 5.1 | 5 | 5 | 1020 | 4.43 | 5 |
| 3 | 1,600 | 1.6 | 1 | 2 | 1600 | 1.39 | 2 |

# Webster's Method

**Example**

Apportion $h = 10$ seats to $n = 3$ states with populations 3,300, 5,100, and 1,600, using the method of Webster.

- Previous Webster critical divisor: $\dfrac{p_k}{a_k - 1/2}$
- Next Webster critical divisor: $\dfrac{p_k}{a_k + 1/2}$

| | | $s = 1{,}000$ | | | Webster $d = 1{,}000$ | | | |
|---|---|---|---|---|---|---|---|---|
| $k$ | $p_k$ | $q_K$ | $\lfloor q_k \rfloor$ | Ham | $\frac{p_k}{a_k+1/2}$ | $\frac{p_k}{a_k-1/2}$ | $q$ | $a_k$ |
| 1 | 3,300 | 3.3 | 3 | 3 | 943 | 1,320 | 3.3 | 3 |
| 2 | 5,100 | 5.1 | 5 | 5 | 927 | 1,133 | 5.1 | 5 |
| 3 | 1,600 | 1.6 | 1 | 2 | 640 | 1,067 | 1.6 | 2 |

# Webster's Method

## Example

Find the Webster apportionments when $n = 4$, $h = 10$, with populations below:

| | | $s = 1{,}000$ | | | | Webster $d = 940$ | | | |
|---|---|---|---|---|---|---|---|---|---|
| $k$ | $p_k$ | $q_K$ | $\lfloor q_k \rfloor$ | Ham | Web | $\frac{p_k}{a_k+1/2}$ | $\frac{p_k}{a_k-1/2}$ | $q$ | $a_k$ |
| 1 | 1,200 | 1.2 | 1 | 1 | 1 | 800 | 2,400 | 1.28 | 1 |
| 2 | 1,400 | 1.4 | 1 | 2 | 1 | 933 | 2,800 | 1.49 | 1 |
| 3 | 2,200 | 2.2 | 2 | 2 | 2 | 880 | 1,467 | 2.34 | 2 |
| 4 | 5,200 | 5.2 | 5 | 5 | 5 | 945 | 1,156 | 5.53 | 6 |
| | 10,000 | | 9 | 10 | 9 | | | | 10 |

# Webster's Method

## Example

Find the Webster apportionments when $n = 4$, $h = 10$, with populations below:

| $p_k$ | 1,200 | 1400 | 2200 | 5,200 |
|---|---|---|---|---|
| 0.5 | 2,400 | 2,800 | 4,400 | 10,400 |
| 1.5 | 800 | 933 | 1467 | 3,467 |
| 2.5 | 480 | 560 | 880 | 2,080 |
| 3.5 | 343 | 400 | 629 | 1,486 |
| 4.5 | 267 | 311 | 489 | 1,155 |
| 5.5 | 218 | 255 | 400 | 945 |
| 6.5 | 185 | 215 | 338 | 800 |

# Divisor Methods So Far

## Divisor Methods

- Choose a modified divisor $d$
- Compute the modified quotas $p_k/d$
- Round them to get $a_k$.
- If $a_1 + a_2 + \cdots + a_n = h$, we have an apportionment.
- Otherwise, choose a new $d$ and try again.

- Jefferson rounds down, favors large states
- Adams round up, favors small states
- Webster rounds "normally", splits the difference
  - Often the same as Hamilton, but not always!
- Are there other options?

# Rounding

## Definition

A rounding function is a function that takes in a real number, outputs an integer, and has the following two properties:

1. If $x$ is an integer, then $f(x) = x$.
2. If $x > y$, then $f(x) \geq f(y)$.

That is:

- Every integer rounds to itself
- Rounding will never take the bigger number and make it smaller.

# Divisor Methods

## Definition

A divisor method is an apportionment method that works as follows.

- Choose a rounding function $f$.
- Choose a modified divisor $d$
- Compute the modified quotas $p_k/d$
- Round these according to our chosen rounding function to obtain $a_k = f(p_k/d)$.
- If $a_1 + a_2 + \cdots + a_n = h$, then we have our apportionment.
- Otherwise, choose a new $d$ and try again.

# Rounding

## Rounding Functions

- Rounding down: $\lfloor x \rfloor$ Jefferson
- Rounding up: $\lceil x \rceil$ Adams
- "Grade school" or arithmetic rounding Webster

## Arithmetic rounding

- Find the number halfway between two integers
- Round up if you're above it, and down if you're below it
- But what do we mean by "half"?

# Rounding

## Discussion Question

- What number is halfway between 2 and 4?
- What number is halfway between 1 and 100?
- What number is halfway between $\frac{1}{2}$ and $\frac{1}{4}$?
- What does "halfway" mean?

# Averages

**Definition**

The arithmetic mean of two numbers $m$ and $n$ is $\frac{m+n}{2}$.

**Definition**

The geometric mean of two numbers $m$ and $n$ is $\sqrt{mn}$.

**Example**

- The arithmetic mean of 2 and 4 is $\frac{2+4}{2} = \frac{6}{2} = 3$
- The geometric mean of 2 and 4 is $\sqrt{2 \cdot 4} = \sqrt{8} = 2.83$
- Both of these are halfway between 2 and 4, for some definition of "halfway".

# Geometric Averages

## Example

- The arithmetic mean of 1 and 100 is $\frac{100+1}{2} = \frac{101}{2} = 50.5$
- The geometric mean of 1 and 100 is $\sqrt{1 \cdot 100} = \sqrt{100} = 10$
- Both of these are halfway between 1 and 100, for some definition of "halfway".

## Geometric mean

- Good when we want to think about ratios
- Good when numbers are very different sizes from each other

# Geometric Rounding

| | | | | | | | |
|---|---|---|---|---|---|---|---|
| 0 to 1 | 1 to 2 | 2 to 3 | 3 to 4 | 4 to 5 | 5 to 6 | 6 to 7 | 7 to 8 |
| $\sqrt{0}$ | $\sqrt{2}$ | $\sqrt{6}$ | $\sqrt{12}$ | $\sqrt{20}$ | $\sqrt{30}$ | $\sqrt{42}$ | $\sqrt{56}$ |
| 0 | 1.414 | 2.449 | 3.464 | 4.472 | 5.477 | 6.481 | 7.483 |

**Definition**

- If a number is between $m$ and $m+1$, we can round geometrically by rounding it up if it's above the geometric mean $\sqrt{m(m+1)}$, and down if it's below the geometric mean.
- Useful cutoffs are given in the table above.

# Hill's Method

**Definition**

Hill's method is the apportionment method that works as follows.

- Choose a modified divisor $d$
- Compute the modified quotas $p_k/d$
- Round these modified quotas geometrically to the geometrically-nearest integer to obtain $a_k$.
- If $a_1 + a_2 + \cdots + a_n = h$, then we have our apportionment.
- Otherwise, choose a new $d$ and try again.

# Hill's Method

## Joseph Hill

- Chief statistician at the Census Bureau (1909–1921)
- Hill's method adopted by Congress in 1941
- We still use it today

# The Harmonic Mean

## Discussion Question

- What number is halfway between $\frac{1}{2}$ and $\frac{1}{4}$?

- Arithmetic: $\frac{1/2+1/4}{2} = \frac{3/4}{2} = \frac{3}{8} = 0.375$
- Geometric: $\sqrt{(1/2)(1/4)} = \sqrt{1/8} = \frac{\sqrt{2}}{4} \approx 0.3535$
- Intuitively nice-feeling: $1/3 \approx 0.333$.

# The Harmonic Mean

## Definition

- The harmonic mean of two numbers is the reciprocal of the average of their reciprocals.
- The harmonic mean of $x$ and $y$ is $\dfrac{1}{\left(\frac{1/x+1/y}{2}\right)}$
- Formula looks awful but it's doing something useful.

## Example

The harmonic mean of $1/2$ and $1/4$ is $\frac{1}{\left(\frac{2+4}{2}\right)} = \frac{1}{3}$.

## A nicer-looking formula

$$\frac{1}{\left(\frac{1/x+1/y}{2}\right)} = \frac{2}{\frac{1}{x}+\frac{1}{y}} = \frac{2}{\frac{y+x}{xy}} = \frac{2xy}{x+y}.$$

# Harmonic Rounding

| | | | | | | | |
|---|---|---|---|---|---|---|---|
| 0 to 1 | 1 to 2 | 2 to 3 | 3 to 4 | 4 to 5 | 5 to 6 | 6 to 7 | 7 to 8 |
| $\frac{0}{1}$ | $\frac{4}{3}$ | $\frac{12}{5}$ | $\frac{24}{7}$ | $\frac{40}{9}$ | $\frac{60}{11}$ | $\frac{84}{13}$ | $\frac{112}{15}$ |
| 0 | 1.333 | 2.400 | 3.429 | 4.444 | 5.455 | 6.462 | 7.467 |

## Definition

- If a number is between $m$ and $m+1$, we can round harmonically by rounding it up if it's above the harmonic mean $\frac{2m(m+1)}{2m+1}$, and down if it's below the harmonic mean.
- Useful cutoffs are given in the table above.

# Dean's Method

**Definition**

Dean's method is the apportionment method that works as follows.

- Choose a modified divisor $d$
- Compute the modified quotas $p_k/d$
- Round these modified quotas harmonically to the harmonically-nearest integer to obtain $a_k$.
- If $a_1 + a_2 + \cdots + a_n = h$, then we have our apportionment.
- Otherwise, choose a new $d$ and try again.

- James Dean was a math professor at the University of Vermont in the early 1800s
- Dean's method was considered in 1830, but never used.

# Comparison of rounding methods

| | Rounding Function and Method | | | | |
|---|---|---|---|---|---|
| | Round Up | Harmonic | Geometric | Arithmetic | Round Down |
| | Adams | Dean | Hill | Webster | Jefferson |
| 0–1 | 0 | 0 | 0 | 0.5 | 1 |
| 1–2 | 1 | 1.333 | 1.414 | 1.5 | 2 |
| 2–3 | 2 | 2.400 | 2.449 | 2.5 | 3 |
| 3–4 | 3 | 3.429 | 3.464 | 3.5 | 4 |
| 4–5 | 4 | 4.444 | 4.472 | 4.5 | 5 |
| 5–6 | 5 | 5.455 | 5.477 | 5.5 | 6 |
| 6–7 | 6 | 6.462 | 6.481 | 6.5 | 7 |
| 7–8 | 7 | 7.467 | 7.484 | 7.5 | 8 |

**Example**

Apportion $h = 10$ seats to $n = 3$ states with populations 1,385, 2,390, and 6,225, using Hamilton, Webster, Hill, and Dean.

| | | $s = 1{,}000$ | | | $d = 1{,}000$ | | | |
|---|---|---|---|---|---|---|---|---|
| $k$ | $p_k$ | $q_K$ | $\lfloor q_k \rfloor$ | Ham | Quota | Webster | Hill | Dean |
| 1 | 1,385 | 1.385 | 1 | 1 | 1.385 | 1 | 1 | 2 |
| 2 | 2,390 | 2.390 | 2 | 3 | 2.390 | 2 | 2 | 2 |
| 3 | 6,225 | 6.225 | 6 | 6 | 6.225 | 6 | 6 | 6 |
| | 10,000 | | 9 | 10 | | 9 | 9 | 10 ✓ |