Criteria for Evaluating Divisor Methods

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Divisor Methods

Definition

A divisor method is an apportionment method that works as follows.

- Choose a rounding function f.
- Choose a modified divisor d
- Compute the modified quotas p_k/d
- Round these according to our chosen rounding function to obtain $a_k = f(p_k/d)$.
- If $a_1 + a_2 + \cdots + a_n = h$, then we have our apportionment.
- Otherwise, choose a new d and try again.

Comparison of rounding methods

	Rounding Function and Method				
	Round Up	Harmonic	Geometric	Arithmetic	Round Down
	Adams	Dean	Hill	Webster	Jefferson
0–1	0	0	0	0.5	1
1–2	1	1.333	1.414	1.5	2
2–3	2	2.400	2.449	2.5	3
3–4	3	3.429	3.464	3.5	4
4–5	4	4.444	4.472	4.5	5
5–6	5	5.455	5.477	5.5	6
6–7	6	6.462	6.481	6.5	7
7–8	7	7.467	7.484	7.5	8

Webster, Dean, and Hill

Example

Apportion $h = 10$ seats to $n = 3$ states with populations 1,385, 2,390, and 6,225, using Hamilton, Webster, Hill, and Dean.

		$s = 1{,}000$			$d = 1{,}000$			
k	p_k	q_K	$\lfloor q_k \rfloor$	Ham	Quota	Webster	Hill	Dean
1	1,385	1.385	1	1	1.385	1	1	2
2	2,390	2.390	2	3	2.390	2	2	2
3	6,225	6.225	6	6	6.225	6	6	6
	10,000		9	10		9	9	10 ✓

Webster, Dean, and Hill

Example

Apportion $h = 10$ seats to $n = 3$ states with populations 1,385, 2,390, and 6,225, using Hamilton, Webster, Hill, and Dean.

		$s = 1{,}000$			$d = 950$			
k	p_k	q_K	$\lfloor q_k \rfloor$	Ham	Quota	Webster	Hill	Dean
1	1,385	1.385	1	1	1.458	1	2	2
2	2,390	2.390	2	3	2.516	3	3	3
3	6,225	6.225	6	6	6.553	7	7	7
	10,000		9	10		11	12	12

- How do find a good d efficiently?

Critical Divisors

When do critical divisors happen?

- Adams and Jefferson: Tipping points at whole numbers.
- Critical divisors at p_k/m.
- Webster: tipping points at exact halves.
- Critical divisors at $\frac{p_k}{m+1/2}$.
- Hill: Tipping points at $\sqrt{m(m+1)}$.
- Critical divisors at $\frac{p_k}{\sqrt{m(m+1)}}$.
- Dean: Tipping points at $\frac{2m(m+1)}{2m+1}$.
- Critical divisors at $\frac{p_k}{\left(\frac{2m(m+1)}{2m+1}\right)} = \frac{p_k(2m+1)}{2m(m+1)}$.

Critical Divisors

Method	Critical divisor for state k
Adams	$\frac{p_k}{a_k}$
Dean	$\frac{p_k(2(a_k+1))}{2a_k(a_k+1)}$
Hill	$\frac{p_k}{\sqrt{a_k(a_k+1)}}$
Webster	$\frac{p_k}{(a_k+1/2)}$
Jefferson	$\frac{p_k}{a_k+1}$

- Probably easier to think of list of numbers.
- Hill: $\frac{p_k}{1.414}, \frac{p_k}{2.449}, \frac{p_k}{3.464}, \ldots$
- Dean: $\frac{p_k}{1.333}, \frac{p_k}{2.400}, \frac{p_k}{3.429}, \ldots$

Webster critical divisors

Example

Apportion $h = 10$ seats to $n = 3$ states with populations 1,385, 2,390, and 6,225, using Webster's method.

		$s = 1{,}000$				$d = 957$	
k	p_k	q_K	a_k	$\frac{p_k}{a_k - 1/2}$	$\frac{p_k}{a_k + 1/2}$	Quota	Webste
1	1,385	1.385	1	$\frac{1385}{0.5} = 2770$	$\frac{1385}{1.5} = 923$	1.447	1
2	2,390	2.390	2	$\frac{2390}{1.5} = 1593$	$\frac{2390}{2.5} = 956$	2.497	2
3	6,225	6.225	6	$\frac{6225}{5.5} = 1132$	$\frac{6225}{6.5} = 958$	6.501	7
	10,000		9				10 ✓

Hill critical divisors

Example

Apportion $h = 10$ seats to $n = 3$ states with populations 1,385, 2,390, and 6,225, using Hill's method.

		$s = 1{,}000$				$d = 978$	
k	p_k	q_K	a_k	$\frac{p_k}{\sqrt{a_k(a_k-1)}}$	$\frac{p_k}{\sqrt{a_k(a_k+1)}}$	Quota	Hill
1	1,385	1.385	1	$\frac{1385}{0} = \infty$	$\frac{1385}{1.414} = 979$	1.416	2
2	2,390	2.390	2	$\frac{2390}{1.414} = 1690$	$\frac{2390}{2.449} = 976$	2.444	2
3	6,225	6.225	6	$\frac{6225}{5.477} = 1137$	$\frac{6225}{6.481} = 961$	6.365	6
	10,000		9				10 ✓

Dean critical divisors

Example

Apportion $h = 10$ seats to $n = 3$ states with populations 1,385, 2,390, and 6,225, using Dean's method.

		$s = 1{,}000$				$d = 1{,}000$	
k	p_k	q_K	a_k	$\frac{p_k(2a_k+1)}{2a_k(a_k-1)}$	$\frac{p_k(2a_k+1)}{2a_k(a_k+1)}$	Quota	Dean
1	1,385	1.385	2	$\frac{1385}{1.333} = 1039$	$\frac{1385}{2.400} = 577$	1.385	2
2	2,390	2.390	2	$\frac{2390}{1.333} = 1,793$	$\frac{2390}{2.400} = 996$	2.390	2
3	6,225	6.225	6	$\frac{6225}{5.455} = 1,141$	$\frac{6225}{6.462} = 963$	6.225	6
	10,000		9				10 ✓

A Table of Hill Critical Divisors

m	State 1	State 2	State 3
0	$\frac{1{,}385}{0} = \infty$	$\frac{2{,}390}{0} = \infty$	$\frac{6{,}225}{0} = \infty$
1	$\frac{1{,}385}{1.414} = 979$	$\frac{2{,}390}{1.414} = 1,690$	$\frac{6{,}225}{1.414} = 4,402$
2	$\frac{1{,}385}{2.449} = 565$	$\frac{2{,}390}{2.449} = 976$	$\frac{6{,}225}{2.449} = 2,541$
3	$\frac{1{,}385}{3.464} = 400$	$\frac{2{,}390}{3.464} = 690$	$\frac{6{,}225}{3.464} = 1,797$
4	$\frac{1{,}385}{4.472} = 310$	$\frac{2{,}390}{4.472} = 690$	$\frac{6{,}225}{4.472} = 1,392$
5	$\frac{1{,}385}{5.477} = 253$	$\frac{2{,}390}{5.477} = 436$	$\frac{6{,}225}{5.477} = 1,137$
6	$\frac{1{,}385}{6.481} = 214$	$\frac{2{,}390}{6.481} = 369$	$\frac{6{,}225}{6.481} = 961$

Criteria

- We've learned about many apportionment methods
  - Hamilton's Method
    - Also Lowndes's Method
  - Divisor Methods
    - Jefferson, Adams, Webster, Hill, Dean
- How do we decide which ones are good?

Discussion Question

- What do we want out of an apportionment method?
- Some too obvious to say but we should say them anyway

Neutrality

Definition

An apportionment method is neutral if permuting the populations of states permutes the resulting numbers of seats in the same way.

- Can still have bias towards large or small states
- Can't have a (formal) bias towards Western states, or conservative states, or states whose names start with a vowel

Discussion Question

What aspect of our current apportionment system isn't neutral?

Proportionality

Definition

An apportionment method is proportional if it produces the same result for two censuses with the same house size, and the same relative populations p_k/p.

- Idea:If every state doubles, that shouldn't change the apportionment

Definition

We say the population distribution of a census is the list $p_1/p, p_2/p, \ldots, p_n/p$.

- Tells you what fraction of the total population each state has
- Proportional methods depend only on the population distribution

Proportionality

Proposition

Hamilton's method is proportional.

Proof.

- Hamilton's method depends only on the standard quotas $q_k = p_k/s$.
- The standard divisor is $s = p/h$
- Can rewrite this as $q_k = \frac{p_k}{p/h} = h\frac{p_k}{p}$
- Standard quota depends only on the house size h and the population distribution p_k/p.

□

Proportionality

Proposition

All divisor methods are proportional.

Proof by Modified Divisors.

- Method works by dividing p_k by a divisor d
- If every population increases by a factor of c can increase divisor to cd
- Modified quotas will be $\frac{cp_k}{cd} = \frac{p_k}{d}$.
- Get the same modified quotas with the same rounded results.

$\square$

Proportionality

Proposition

All divisor methods are proportional.

Proof by Critical Divisors.

- Method works by writing down the critical divisors from largest to smallest, and taking the h largest
- Critical divisors are $\frac{p_k}{f(m)}$ for some rounding function m
- Instead can compute $\frac{p_k/p}{f(m)}$
- This just divides our entire list by p without changing the order
- The list, and thus the result, only depends on the population distribution.

□

Order-Preserving

Definition

An apportionment method is order-preserving if, whenever $a_i > a_j$, then $p_i > p_j$.

- Bigger states should never get fewer seats
- We really really want this one
- All our methods so far do have it.

Remark

It would be reasonable and logical to call this the "population monotone" property. But we've already used that term for something else.

Quota Rules

Definition

- We say it's a quota violation if an apportionment method gives a state more representatives than its upper quota, or less than its lower quota.
- An apportionment method satisfies the quota rule if it assigns every state either its lower quota or its upper quota.
- An apportionment method satisfies the upper quota rule if it never assigns a state more than its upper quota.
- A violation of this rule is an upper quota violation.
- An apportionment method satisfies the lower quota rule if it never assigns a state less than its lower quota.
- A violation of this rule is an lower quota violation.

House Monotonicity

Definition

An apportionment method is called house monotone if an increase in h, while all other parameters remain the same, can never cause any seat allocation a_k to decrease.

- Saw in an early example that Hamilton's method is not house monotone
- Alabama paradox in 1880 shows Hamilton's method is not house monotone *in practice*.

Exercise

Lowndes's method is not house monotone.

House Monotonicity

Proposition

All divisor methods are house monotone.

Proof.

- Consider any divisor method with rounding function f
- Suppose divisor d apportions exactly h seats
- If we want to increase h, we will need to decrease d
- Get a larger modified quota p_k/d' for each state
- By definition, rounding a larger number won't give a smaller number
- That is, since $p_k/d' > p_k/d$, we know $f(p_k/d') \geq f(p_k/d)$
- Therefore no state will get a smaller apportionment.

□

House Monotonicity

Proposition

All divisor methods are house monotone.

Corollary

Hamilton's method is not a divisor method.

- More interesting than it sounds
- We didn't describe Hamilton's method as a divisor method
- So what does this tell us?
- We *cannot* describe Hamilton's method with *any* rounding function.

Population Monotonicity

Definition

- A method is called population monotone if a state can never lose a seat when its population increases while no other state's population increases.
- In algebraic terms, whenever $a_i' < a_i$ and $a_j' > a_j$, it must be the case either that $p_i' < p_i$ or $p_j' > p_j$.

Proposition

Hamilton's method isn't population monotone.

Exercise

Lowndes's method isn't population monotone.

Population Monotonicity

Proposition

All divisor methods are population monotone.

Proof.

- Suppose $a_i' < a_i$ and $a_j' > a_j$.
- Want to show: either $p_i' < p_i$ or $p_j' > p_j$.
- (Stop and think about what those algebraic sentences mean.)
- State i's modified quota went down, and State j's went up
- Thus $\frac{p_i'}{d'} < \frac{p_i}{d}$ and $\frac{p_j'}{d'} > \frac{p_j}{d}$.
- Rearrange: $p_i' < p_i \frac{d'}{d}$ and $p_j' > p_j \frac{d'}{d}$.

Population Monotonicity

Proposition

All divisor methods are population monotone.

Proof.

- Suppose $a_i' < a_i$ and $a_j' > a_j$.
- Want to show: either $p_i' < p_i$ or $p_j' > p_j$.
- We know $p_i' < p_i \frac{d'}{d}$ and $p_j' > p_j \frac{d'}{d}$.
- If $d' < d$ then $\frac{d'}{d} < 1$ so $p_i' < p_i \frac{d'}{d} < p_i$.
- If $d' > d$ then $\frac{d'}{d} > 1$ so $p_j' > p_j \frac{d'}{d} > p_j$.
- Either way, that's what we want to prove.

□