More on Criteria for Evaluating Divisor Methods

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Apportionment method criteria

- An apportionment method is neutral if permuting the populations of states permutes the resulting numbers of seats in the same way.
- An apportionment method is proportional if it produces the same result for two censuses with the same house size, and the same relative populations p_k/p.
- An apportionment method is order-preserving if, whenever $a_i > a_j$, then $p_i > p_j$.
- All our methods so far satisfy these criteria
- Useful in checklists, and in proofs

Apportionment method criteria

Definition

- We say it's a quota violation if an apportionment method gives a state more representatives than its upper quota, or less than its lower quota.
- An apportionment method satisfies the quota rule if it assigns every state either its lower quota or its upper quota.
- We can also talk about the upper quota rule and lower quota rule.
- Hamilton's method satisfies the quota rule
- Jefferson's and Adams's methods do not
- What about the others?

Apportionment method criteria

- An apportionment method is called house monotone if an increase in h, while all other parameters remain the same, can never cause any seat allocation a_k to decrease.
- A method is called population monotone if a state can never lose a seat when its population increases while no other state's population increases.
  - In algebraic terms, whenever $a_i' < a_i$ and $a_j' > a_j$, it must be the case either that $p_i' < p_i$ or $p_j' > p_j$.
- Hamilton's and Lowndes's methods aren't house or population monotone
- All divisor methods are house and population monotone.
- Are the two ideas related?

Population Monotonicity and House Monotonicity

Proposition

Any method that is population monotone is also house monotone.

Proof.

- Suppose a method is population monotone
- What happens when house size changes from h to $h+1$?
- Assume no populations change, so $p_i' = p_i$ and $p_j' = p_j$.
- At least one state will gain a seat. We assume $a_j' > a_j$.
- Imagine some other state loses a seat, so $a_i' < a_i$.
- By population monotone, we'd need either $p_i' < p_i$ or $p_j' > p_j$
- But neither thing is true, so that's not possible.

□

Population Monotonicity and Order-Preservingness

Proposition

Any method that is population monotone and neutral must be order-preserving.

Proof.

- Suppose a method is neutral and monotone, and $p_j > p_i$.
- Imagine a new census swapping populations of states i and j
- That is, $p_i' = p_j$ and $p_j' = p_i$, and then $p_k' = p_k$.
- By population monotonicity, $a_i' \geq a_i$ or $a_j' \leq a_j$. (Or both!)
- By neutrality, $a_i' = a_j$ and $a_j' = a_i$.
- Either $a_j = a_i' \geq a_i$, or $a_i = a_j' \leq a_j$.
- Either way, $a_i \leq a_j$, as needed.

□

The New States Paradox

Definition

- Suppose state k is joining the union as a new state, and thus $p_k = 0$ and $p'_k > 0$.
- Suppose there are other states i and j whose populations are unchanged.
- We say a *new states paradox* or *Oklahoma paradox* occurs if $a'_i < a_i$ and $a'_j > a_j$.

- This violates population monotonicity
- No divisor method experiences the new states paradox
- Hamilton's method can experience the new states paradox.

Two Approaches to Population Change

Absolute Change

- How many more or fewer people?
- Gives a number
- $\Delta p_k = p'_k - p_k$

Relative Change

- What is the growth rate?
- Gives a fraction/percentage
- $\frac{\Delta p_k}{p_k} = \frac{p'_k - p_k}{p_k}$

Absolute and Relative Population Change

k	1	2	3
p_k	10,000	10,000	100,000
p'_k	12,000	20,000	111,000
Δp_k	2,000	10,000	11,000
$\Delta p_k / p_k$	0.2	1.0	0.11
%	20%	100%	11%

Discussion Question

- Which state grew the fastest?
- Which state grew the most?

Relative Population Monotonicity

Definition

An apportionment method is relative population monotone if,

- When we consider states with positive population,
- whenever $a_i' < a_i$ and $a_j' > a_j$,
- then $\frac{\Delta p_j}{p_j} > \frac{\Delta p_i}{p_i}$.

Discussion Question

- What does it mean to not have positive population?
- What issue have we talked about that involves non-positive populations?

Relative and Absolute Population Monotonicity

Proposition

If an apportionment method is relative population monotone, then it is population monotone.

Proof.

- Suppose $a_i' < a_i$ and $a_j' > a_j$, and all populations are positive.
- Then $\frac{\Delta p_j}{p_j} > \frac{\Delta p_i}{p_i}$.
- In particular, either $\frac{\Delta p_j}{p_j} > 0$ or $\frac{\Delta p_i}{p_i} < 0$. (Or both!)
- If $\frac{\Delta p_j}{p_j} > 0$ then $\Delta p_j > 0$ so $p_j' > p_j$.
- If $\frac{\Delta p_i}{p_i} < 0$ then $\Delta p_i < 0$ so $p_i' < p_i$.
- Thus the method is (absolute) population monotone.

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Divisor Methods and Relative Population Monotonicity

Proposition

All divisor methods are relative population monotone.

Proof.

- Suppose $a_i' < a_i$ and $a_j' > a_j$.
- Want to show: $\frac{\Delta p_j}{p_j} > \frac{\Delta p_i}{p_i}$.
- Know that $\frac{p_i'}{d'} < \frac{p_i}{d}$ and $\frac{p_j'}{d'} > \frac{p_j}{d}$.
- Rearranging: $\frac{p_i'}{p_i} < \frac{d'}{d}$ and $\frac{p_j'}{p_j} > \frac{d'}{d}$.
- Combining: $\frac{p_j'}{p_j} > \frac{p_i'}{p_i}$.
- How do we interpret that? State i has grown slower (or shrunk faster) than State j.

Divisor Methods and Relative Population Monotonicity

Proposition

All divisor methods are relative population monotone.

Proof.

- Want to show: $\frac{\Delta p_j}{p_j} > \frac{\Delta p_i}{p_i}$.
- We know: $\frac{p_j'}{p_j} > \frac{p_i'}{p_i}$.
- Subtract 1: $\frac{p_j'}{p_j} - 1 > \frac{p_i'}{p_i} - 1$
- $\frac{p_j' - p_j}{p_j} > \frac{p_i' - p_i}{p_i}$
- That's what we wanted to prove!

Relative and Absolute Population Monotonicity

- If an apportionment method is relative population monotone, then it is population monotone.
- Converse is not true: possible to be absolute population monotone and not relative population monotone
- We say that "relative population monotone" is stronger than "absolute population monotone"
- But it's not very much stronger!

Proposition

If an apportionment method is population monotone and proportional, then it's relative population monotone.

Relative and Absolute Population Monotonicity

Proposition

If an apportionment method is population monotone and proportional, then it's relative population monotone.

Proof.

- What would have to not be relative population monotone?
- Need $a_i' < a_i$ and $a_j' > a_j$
- And also $\frac{\Delta p_j}{p_j} \leq \frac{\Delta p_i}{p_i}$.
- Set $r = 1 + \frac{\Delta p_i}{p_i} = \frac{p_i'}{p_i}$
- Set $s = 1 + \frac{\Delta p_j}{p_j} = \frac{p_j'}{p_j}$.
- $0 < s$ since $0 < p_j'$
- $s \leq r$ by hypothesis.

Relative and Absolute Population Monotonicity

Proposition

If an apportionment method is population monotone and proportional, then it's relative population monotone.

Proof.

- $r = \frac{p'_i}{p_i}$ and $s = \frac{p'_j}{p_j}$
- $0 < s \leq r$.
- Imagine third census scaled up from second: $p''_k = \frac{p'_k}{r}$.
- By proportionality, $a''_k = a'_k$.
- $p''_i = p'_i/r = \frac{p'_i}{p'_i/p_i} = p_i$
- $p''_j = p'_j/r = \frac{p_j s}{r} \leq p_j$.

Relative and Absolute Population Monotonicity

Proposition

If an apportionment method is population monotone and proportional, then it's relative population monotone.

Proof.

- What do we know?
- $a_i'' = a_i' < a_i$ and $a_j'' = a_j' > a_j$
- $p_i'' = p_i$ and $p_j'' \leq p_j$.
- That violates absolute population monotonicity.

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- We only want to think about proportional methods
- We can treat absolute and relative population monotonicity as the same.

Criteria Summary

- Hamilton's method:
    - Satisfies quota rule
    - Isn't house monotone
    - Isn't population monotone
- Divisor methods:
    - Are house monotone
    - Are population monotone
    - Can they satisfy the quota rule?
    - We know that Jefferson and Adams violate quota
- Can we find a method that satisfies the quota rule, while avoiding the paradoxes of Hamilton's method?

An Impossibility Theorem

Theorem (Balinski and Young)

No apportionment rule that is neutral and population monotone can satisfy the quota rule.

Proof.

- Want to show something *can* happen
- Need a counterexample
- We'll construct a pair of censuses where you cannot be neutral and satisfy the quota rule without violating population monotonicity.

An Impossibility Theorem

Theorem (Balinski and Young)

No apportionment rule that is neutral and population monotone can satisfy the quota rule.

Proof.

- Allocate ten seats to the following two censuses:

$$\begin{aligned} p_1 &= 69,900 & p_1' &= 68,000 \\ p_2 &= 5,200 & p_2' &= 5,500 \\ p_3 &= 5,000 & p_3' &= 5,600 \\ p_4 &= 19,900 & p_4' &= 5,700. \end{aligned}$$

An Impossibility Theorem

Theorem (Balinski and Young)

No apportionment rule that is neutral and population monotone can satisfy the quota rule.

Proof.

k	p_k	q_k	$\lfloor q_k \rfloor$	$\lceil q_k \rceil$
1	69,900	6.99	6	7
2	5,200	0.52	0	1
3	5,000	0.50	0	1
4	19,900	1.99	1	2

- State 1 gets at most 7
- State 4 gets at most 2
- Either State 2 or State 3 has to get one
- By order-preserving, State 2 has to get at least one seat.

An Impossibility Theorem

Theorem (Balinski and Young)

No apportionment rule that is neutral and population monotone can satisfy the quota rule.

Proof.

k	p_k	q_k	$\lfloor q_k \rfloor$	$\lceil q_k \rceil$
1	68,000	8.02	8	9
2	5,500	0.65	0	1
3	5,600	0.66	0	1
4	5,700	0.67	0	1

- State 1 gets at least 8
- At most two of the other states get seats
- By order preserving, State 2 can't get a seat.

An Impossibility Theorem

Theorem (Balinski and Young)

No apportionment rule that is neutral and population monotone can satisfy the quota rule.

Proof.

$$\begin{aligned} p_1 &= 69,900 & p_1' &= 68,000 \\ p_2 &= 5,200 & p_2' &= 5,500 \\ p_3 &= 5,000 & p_3' &= 5,600 \\ p_4 &= 19,900 & p_4' &= 5,700. \end{aligned}$$

- $a_1 \leq 7$ and $a_1' \geq 8$
- $a_2 = 1$ but $a_2' = 0$
- But $p_1 > p_1'$ and $p_2 < p_2'$
- Violates population monotonicity.

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An Impossibility Theorem

Theorem (Balinski and Young)

No apportionment rule that is neutral and population monotone can satisfy the quota rule.

Corollary

No divisor method can satisfy the quota rule.

- Can't satisfy quota and population monotone
- But what about house monotone?

Balinski and Young's Method

Definition (Inductive)

- If $h = 0$, then set $a_k = 0$ for every k.
- Suppose we have an apportionment for some fixed h, given by $a_1, a_2, \ldots, a_n$ such that $a_1 + \cdots + a_n = h$.
- Compute $\frac{p_k}{a_k+1}$ and call this the strength of State k's claim for the next seat.
- We "want" to give the next seat to the state with the strongest claim, but don't want any upper quota violations.
- A state is eligible if $a_k + 1 \leq \left\lceil (h+1)\frac{p_k}{p} \right\rceil$, so that giving the state another seat would not give an upper quota violation.
- Assign seat $h+1$ to the eligible state with strongest claim.

Balinski and Young's Method

Discussion Question

Why do we use Jefferson rather than Adams or Hill or Webster as the base for this method?

Properties of the method

- Avoids lower quota violations because Jefferson
- Avoids upper quota violations by definition
- Therefore, not population monotone
- But obviously house monotone because we add seats one by one

Discussion Question

What could potentially go wrong?

Balinski and Young's Method

Proposition

At each inductive stage of the method of Balinski and Young, at least one state is eligible to receive the next seat.

Idea of proof.

- State is ineligible when it has too many seats
- Needs to have a lot of seats relative to h
- Not every state can have more than their share of seats at once!

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Balinski and Young's Method

Proposition

At each inductive stage of the method of Balinski and Young, at least one state is eligible to receive the next seat.

Proof.

- Suppose we've apportioned h seats
- After we apportion $h+1$, will have $s = \frac{p}{h+1}$
- Standard quotas will be $q_k = \frac{p_k}{s} = (h+1)\frac{p_k}{p}$
- State k is ineligible if $a_k + 1 > \lceil q_k \rceil$.
- Both whole numbers: this can only happen if $a_k \geq q_k$.

Balinski and Young's Method

Proposition

At each inductive stage of the method of Balinski and Young, at least one state is eligible to receive the next seat.

Proof.

- State k is ineligible if $a_k \geq q_k = (h+1)\frac{p_k}{p}$
- Imagine all states are ineligible

$$\begin{aligned} a_1 + a_2 + \cdots + a_n &\geq (h+1)\frac{p_1}{p} + (h+1)\frac{p_2}{p} + \cdots + (h+1)\frac{p_n}{p} \\ &= \frac{h+1}{p}\left(p_1 + p_2 + \cdots + p_n\right) \\ &= \frac{h+1}{p}(p) = h+1 \end{aligned}$$

Balinski and Young's Method

Proposition

At each inductive stage of the method of Balinski and Young, at least one state is eligible to receive the next seat.

Proof.

- If all states are ineligible, then $a_1 + \cdots + a_n \geq h + 1$
- But we know $a_1 + \cdots + a_n = h$
- So at least one state is eligible.

$\square$

Balinski and Young's Method

h	Jefferson Critical Divisor			Jefferson Apportionment		
	$\frac{p_1}{a_1+1}$	$\frac{p_2}{a_2+1}$	$\frac{p_3}{a_3+1}$	a_1	a_2	a_3
0				0	0	0
1	7	22	**71**	0	0	1
2	7	22	**35.5**	0	0	2
3	7	22	**23.67**	0	0	3
4	7	**22**	17.75	0	1	3
5	7	11	**17.75**	0	1	4
6	7	11	**14.2**	0	1	5
7	7	11	**11.83**	0	1	6
8	7	**11**	10.14	0	2	6
9	7	7.33	**10.14**	0	2	7
10	7	7.33	**8.875**	0	2	8
11	7	7.33	**7.89**	0	2	9
12	7	**7.33**	7.1	0	3	9
13	7	5.5	**7.1**	0	3	10
14	**7**	5.5	6.45	1	3	10

Balinski and Young's Method

h	Jefferson Critical Divisor			Jefferson Apportionment			Standard Quotas			Balinski and Young Apportionment		
	$\frac{p_1}{a_1+1}$	$\frac{p_2}{a_2+1}$	$\frac{p_3}{a_3+1}$	a_1	a_2	a_3	q_1	q_2	q_3	a_1	a_2	a_3
0				0	0	0						
1	7	22	**71**	0	0	1	0.07	0.22	0.71	0	0	1
2	7	22	**35.5**	0	0	2	0.14	0.44	1.42	0	0	2
3	7	22	**23.67**	0	0	3	0.21	0.66	2.13	0	0	3
4	7	**22**	~~17.75~~	0	1	3	0.28	0.88	.84	0	1	3
5	7	11	**17.75**	0	1	4	0.35	1.1	3.55	0	1	4
6	7	11	**14.2**	0	1	5	0.42	1.32	4.26	0	1	5
7	7	**11**	~~11.83~~	0	1	6	0.49	1.54	~~4.97~~	0	2	5
8	7	7.33	**11.83**	0	2	6	0.56	1.76	5.68	0	2	6
9	7	7.33	**10.14**	0	2	7	0.63	1.98	6.39	0	2	7
10	7	7.33	**8.88**	0	2	8	0.7	2.2	7.1	0	2	8
11	7	**7.33**	~~7.89~~	0	2	9	0.77	2.42	~~7.81~~	0	3	8
12	7	5.5	**7.89**	0	3	9	0.84	2.64	8.52	0	3	9
13	7	5.5	**7.1**	0	3	10	0.91	2.86	9.23	0	3	10