Which Divisor Methods Are Best?

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Criteria Summary

- Hamilton's method:
  - Satisfies quota rule
  - Isn't house monotone
  - Isn't population monotone
- Divisor methods:
  - Are house monotone
  - Are population monotone
  - Can they satisfy the quota rule?
  - We know that Jefferson and Adams violate quota
- Can we find a method that satisfies the quota rule, while avoiding the paradoxes of Hamilton's method?

An Impossibility Theorem

Theorem (Balinski and Young)

No apportionment rule that is neutral and population monotone can satisfy the quota rule.

Corollary

No divisor method can satisfy the quota rule.

- Can't satisfy quota and population monotone
- But *can* satisfy quota and house monotone.

Comparing Divisor Methods

Our Divisor Methods

- Jefferson: Round Down
  - Favors large states
- Adams: Round up
  - Favors small states
- Webster: Arithmetic mean
- Hill: Geometric mean
- Dean: Harmonic mean

- Jefferson and Adams: biased and frequently violate quota
- How to choose among the other three?

Comparing Divisor Methods

History

- Early 1900s: contentious fights over size of h
- 1920: No reapportionment at all!
- Big debate between Webster's method and Hill's method
- 1930s: Webster and Hill agreed, so they didn't need to pick
- 1940s: Webster and Hill almost agreed
  - Webster gave one extra seat to Michigan
  - Hill gave that seat to Arkansas
  - Arkansas leaned Democratic and Michigan leaned Republican
  - Congress also leaned Democratic
  - Established Hill as our method going forward.

- Was that a good choice or a bad choice?

Why Webster?

What does Webster accomplish?

- $\frac{p_i}{d} - 1/2 < a_i \leq \frac{p_i}{d} + 1/2$
- $\frac{p_i}{a_i + 1/2} < d \leq \frac{p_i}{a_i - 1/2}$
- (These are just the critical divisors for state i!)
- Since d is the same for all states, $\frac{p_j}{a_j + 1/2} < d \leq \frac{p_i}{a_i - 1/2}$
- Take reciprocals: $\frac{a_i - 1/2}{p_i} < \frac{a_j + 1/2}{p_j}$
- Multiply by 2: $\frac{a_i + a_i - 1}{p_i} < \frac{a_j + a_j + 1}{p_j}$
- Separate and rearrange: $\frac{a_i}{p_i} - \frac{a_j}{p_j} < \frac{a_j + 1}{p_j} - \frac{a_i - 1}{p_j}$.

Why Webster?

What does Webster accomplish?

- $\frac{a_i}{p_i} - \frac{a_j}{p_j} < \frac{a_j+1}{p_j} - \frac{a_i-1}{p_j}$.
- Can we interpret that?
- $\frac{a_i}{p_i}$ is the number of seats divided by the number of people
- Fraction of a seat "belonging" to each person in the state
- In the US, this number is relatively close to $\frac{1}{700{,}000}$.

Definition

The degree of representation of a state k is the number $\frac{a_k}{p_k}$, which measures the fraction of a congressional seat each individual citizen is allocated.

Why Webster?

Definition

The degree of representation of a state k is the number $\frac{a_k}{p_k}$, which measures the fraction of a congressional seat each individual citizen is allocated.

Example

- Suppose $p_1 = 10$, $p_2 = 25$, $h = 5$.
- If $a_1 = 2$ and $a_2 = 3$ then we have $\frac{2}{10} = 0.2$ and $\frac{3}{25} = 0.12$.
  - State 1 is overrepresented
- If $a_1 = 1$ and $a_2 = 4$ then we have $\frac{1}{10} = 0.1$ and $\frac{4}{25} = 0.16$.
  - State 2 is overrepresented
- Which is less unfair?
- Idea: $0.2 - 0.12 = 0.08$, but $0.16 - 0.1 = 0.06$. Choose $(1, 4)$.

Why Webster?

Definition

The degree of representation of a state k is the number $\frac{a_k}{p_k}$, which measures the fraction of a congressional seat each individual citizen is allocated.

What does Webster accomplish?

- In a perfect world, every state would have the same degree of representation
- $\frac{a_i}{p_i} - \frac{a_j}{p_j}$ measures how far apart two states' degrees of representation are
- We'd like this to be as small as possible.
- With Webster, we saw $\frac{a_i}{p_i} - \frac{a_j}{p_j} < \frac{a_j + 1}{p_j} - \frac{a_i - 1}{p_j}$.

Why Webster?

Definition

The degree of representation of a state k is the number $\frac{a_k}{p_k}$, which measures the fraction of a congressional seat each individual citizen is allocated.

What does Webster accomplish?

- $\dfrac{a_i}{p_i} - \dfrac{a_j}{p_j} < \dfrac{a_j + 1}{p_j} - \dfrac{a_i - 1}{p_j}$.
- Left-hand side: how far apart two states are with Webster
- Right-hand side: how far apart they would be if we move a seat from state i to state j.
- If you start with Webster, you can never shrink this gap by moving a seat from one state to another.

Why Webster?

Proposition

Webster's method is the unique apportionment method such that the difference between the degrees of representation of any two states cannot be decreased by transferring a seat from the better represented state to the worse represented state.

- If we ever have a non-Webster apportionment, we can swap a seat to shrink the gap in degree of representation.
- Webster minimizes the gap in degree of representation
- So that makes it best, right?

Why Dean?

$$\frac{2(a_i-1)a_i}{2(a_i-1)+1} \leq \frac{p_i}{d} < \frac{2a_i(a_i+1)}{2a_i+1}$$

$$\frac{2(a_i+1)}{2a_i(a_i+1)} < \frac{d}{p_i} \leq \frac{2a_i-1}{2(a_i-1)a_i}$$

$$p_i\left(\frac{2(a_i+1)}{2a_i(a_i+1)}\right) < d \leq p_i\left(\frac{2a_i-1}{2(a_i-1)a_i}\right)$$

$$p_j\left(\frac{2(a_j+1)}{2a_j(a_j+1)}\right) < d \leq p_i\left(\frac{2a_i-1}{2(a_i-1)a_i}\right)$$

$$p_j\left(\frac{2(a_j+1)}{a_j(a_j+1)}\right) < d \leq p_i\left(\frac{2a_i-1}{(a_i-1)a_i}\right)$$

$$p_j\left(\frac{1}{a_j}+\frac{1}{a_j+1}\right) < p_i\left(\frac{1}{a_i}+\frac{1}{a_i-1}\right)$$

$$\frac{p_j}{a_j}-\frac{p_i}{a_i} < \frac{p_i}{a_i-1}-\frac{p_j}{a_j+1}$$

Why Dean?

What does Dean accomplish

- $\frac{p_j}{a_j} - \frac{p_i}{a_i} < \frac{p_i}{a_i - 1} - \frac{p_j}{a_j + 1}$
- Can we interpret that?
- $\frac{p_j}{a_j}$ is the number of people per seat.
- Size of each congressional district in state j
- In the US, this number is close to 700,000.

Proposition

Dean's method is the unique apportionment method such that the difference between the sizes of districts in any two states cannot be made smaller by transferring a seat from the smaller-district state to the bigger-district state.

Dean versus Webster

Proposition

Dean's method is the unique apportionment method such that the difference between the sizes of districts in any two states cannot be made smaller by transferring a seat from the smaller-district state to the bigger-district state.

Proposition

Webster's method is the unique apportionment method such that the difference between the degrees of representation of any two states cannot be decreased by transferring a seat from the better represented state to the worse represented state.

- Aren't those measuring the same thing?

Dean versus Webster

Example

- Suppose $p_1 = 10, p_2 = 25, h = 5$
- If $a_1 = 2$ and $a_2 = 3$ then
  - Degrees of representation are $2/10 = 0.2$ and $3/25 = 0.12$
  - Difference is 0.08
  - District sizes are $10/2 = 5$ and $25/3 = 8.3$
  - Difference is 3.3
- If $a_1 = 1$ and $a_2 = 4$ then $10/1 = 10$ and $25/4 = 6.25$
  - Degrees of representation are $1/10 = 0.1$ and $4/25 = 0.16$
  - Difference is 0.06
  - District sizes are $10/1 = 10$ and $25/4 = 6.25$
  - Difference is 3.75
- Webster chooses $(1, 4)$, but Dean chooses $(2, 3)$.

Dean versus Webster

Example

- Suppose $p_1 = 10, p_2 = 25, h = 5$
- $a_1 = 2, a_2 = 3$ closer district size
- $a_1 = 1, a_2 = 4$ closer degree of representation
- They're not the same thing!

- Dean minimizes difference in district size
- Webster minimizes difference in the reciprocal
- Dean more favorable to small states, and Webster more favorable to large states
- Can we...split the difference?

Why Hill?

$$\sqrt{a_i(a_i-1)} \leq \frac{p_i}{d} < \sqrt{a_i(a_i+1)}$$

$$\frac{p_i}{\sqrt{a_i(a_i+1)}} < d \leq \frac{p_i}{\sqrt{a_i(a_i-1)}}$$

$$\frac{p_j}{\sqrt{a_i(a_i+1)}} < d \leq \frac{p_i}{\sqrt{a_i(a_i-1)}}$$

$$\frac{p_j^1}{a_j(a_j+1)} < \frac{p_i^2}{a_i(a_i-1)}$$

$$\frac{p_j/a_j}{p_i/a_i} < \frac{p_i/(a_i-1)}{p_j/(a_j+1)}$$

$$\frac{a_i/p_i}{a_j/p_j} < \frac{(a_j+1)/p_j}{(a_i-1)/p_i}$$

Why Hill?

What does Hill do?

$$\frac{p_j/a_j}{p_i/a_i} < \frac{p_i/(a_i-1)}{p_j/(a_j+1)} \tag{1}$$

$$\frac{a_i/p_i}{a_j/p_j} < \frac{(a_j+1)/p_j}{(a_i-1)/p_i} \tag{2}$$

- How do we interpret this?
- (1) says Hill minimizes the *ratio* of district sizes
- (2) says Hill minimizes the ratio of degrees of representation
- The ratios are equivalent, even though the absolute differences are not.

Why Hill?

Example

- Suppose $p_1 = 10, p_2 = 25, h = 5$
- If $a_1 = 2$ and $a_2 = 3$ then
  - Degrees of representation are $2/10 = 0.2$ and $3/25 = 0.12$
  - Ratio is $\frac{2/10}{3/25} = \frac{50}{30} = \frac{5}{3} = 1.\overline{66}$
  - District sizes are $10/2 = 5$ and $25/3 = 8.3$
  - Ratio is $\frac{25/3}{10/2} = \frac{50}{30} = \frac{5}{3} = 1.\overline{66}$.
- If $a_1 = 1$ and $a_2 = 4$ then $10/1 = 10$ and $25/4 = 6.25$
  - Degrees of representation are $1/10 = 0.1$ and $4/25 = 0.16$
  - Ratio is $\frac{4/25}{1/10} = \frac{40}{25} = \frac{8}{5} = 1.6$
  - District sizes are $10/1 = 10$ and $25/4 = 6.25$
  - Ratio is $\frac{10/1}{25/4} = \frac{40}{25} = \frac{8}{5} = 1.6$
- Hill will choose $(1, 4)$.

Why Hill?

Proposition

- *Hill's method is the unique apportionment method such that the ratio between the average sizes of districts in any two states (expressed as number greater than 1) cannot be made smaller by transferring a seat from the smaller-district state to the bigger-district state.*
- *Further, it is also the unique apportionment method such that the ratio between the degrees of representation in any two states cannot be made smaller by transferring a seat from the better-represented state to the worse-represented state.*

Dean, Webster, and Hill

- Dean minimizes absolute gap in district size
- Webster minimizes absolute gap in degree of representation
- Hill minimizes *relative* difference in both district size and degree of representation

- Dean is most favorable to small states
- Webster is most favorable to large states
- Hill somewhere in the middle

- All three can violate quota for some data
- None have ever violated quota on actual census data
- Simulations suggest Webster is least likely to violate quota

My takeaways

- No method is perfect
- We saw three different definitions of fair
- Math can tell you which method is optimal, given your definition of fairness
- Math can't tell you which version of fairness you care about.
- But it can help you think about the implications of your choice.

Test on Tuesday

Test Rules

- Plan to take the whole class
- Will have 6-7 questions
- Bring a one-sided handwritten note sheet
- You can bring a calculator, and should

Test Topics

- Hamilton's method and Lownde's method
- Divisor Methods and Critical Divisors
- Short proofs relating criteria
- Understanding criteria and which methods satisfy them
- Interpreting simple algebraic formulas

Balinski and Young's Method

Definition (Inductive)

- If $h = 0$, then set $a_k = 0$ for every k.
- Suppose we have an apportionment for some fixed h, given by $a_1, a_2, \ldots, a_n$ such that $a_1 + \cdots + a_n = h$.
- Compute $\frac{p_k}{a_k+1}$ and call this the strength of State k's claim for the next seat.
- We "want" to give the next seat to the state with the strongest claim, but don't want any upper quota violations.
- A state is eligible if $a_k + 1 \leq \left\lceil (h+1)\frac{p_k}{p} \right\rceil$, so that giving the state another seat would not give an upper quota violation.
- Assign seat $h+1$ to the eligible state with strongest claim.

Balinski and Young's Method

Discussion Question

Why do we use Jefferson rather than Adams or Hill or Webster as the base for this method?

Properties of the method

- Avoids lower quota violations because Jefferson
- Avoids upper quota violations by definition
- Therefore, not population monotone
- But obviously house monotone because we add seats one by one

Discussion Question

What could potentially go wrong?

Balinski and Young's Method

Proposition

At each inductive stage of the method of Balinski and Young, at least one state is eligible to receive the next seat.

Idea of proof.

- State is ineligible when it has too many seats
- Needs to have a lot of seats relative to h
- Not every state can have more than their share of seats at once!

□

Balinski and Young's Method

Proposition

At each inductive stage of the method of Balinski and Young, at least one state is eligible to receive the next seat.

Proof.

- Suppose we've apportioned h seats
- After we apportion $h+1$, will have $s = \frac{p}{h+1}$
- Standard quotas will be $q_k = \frac{p_k}{s} = (h+1)\frac{p_k}{p}$
- State k is ineligible if $a_k + 1 > \lceil q_k \rceil$.
- Both whole numbers: this can only happen if $a_k \geq q_k$.

Balinski and Young's Method

Proposition

At each inductive stage of the method of Balinski and Young, at least one state is eligible to receive the next seat.

Proof.

- State k is ineligible if $a_k \geq q_k = (h+1)\frac{p_k}{p}$
- Imagine all states are ineligible

$$
\begin{aligned}
a_1 + a_2 + \cdots + a_n &\geq (h+1)\frac{p_1}{p} + (h+1)\frac{p_2}{p} + \cdots + (h+1)\frac{p_n}{p} \\
&= \frac{h+1}{p}(p_1 + p_2 + \cdots + p_n) \\
&= \frac{h+1}{p}(p) = h+1
\end{aligned}
$$

Balinski and Young's Method

Proposition

At each inductive stage of the method of Balinski and Young, at least one state is eligible to receive the next seat.

Proof.

- If all states are ineligible, then $a_1 + \cdots + a_n \geq h + 1$
- But we know $a_1 + \cdots + a_n = h$
- So at least one state is eligible.

□

Balinski and Young's Method

h	Jefferson Critical Divisor			Jefferson Apportionment		
	$\frac{p_1}{a_1+1}$	$\frac{p_2}{a_2+1}$	$\frac{p_3}{a_3+1}$	a_1	a_2	a_3
0				0	0	0
1	7	22	**71**	0	0	1
2	7	22	**35.5**	0	0	2
3	7	22	**23.67**	0	0	3
4	7	**22**	17.75	0	1	3
5	7	11	**17.75**	0	1	4
6	7	11	**14.2**	0	1	5
7	7	11	**11.83**	0	1	6
8	7	**11**	10.14	0	2	6
9	7	7.33	**10.14**	0	2	7
10	7	7.33	**8.875**	0	2	8
11	7	7.33	**7.89**	0	2	9
12	7	**7.33**	7.1	0	3	9
13	7	5.5	**7.1**	0	3	10
14	**7**	5.5	6.45	1	3	10

Balinski and Young's Method

h	Jefferson Critical Divisor			Jefferson Apportionment			Standard Quotas			Balinski and Young Apportionment		
	$\frac{p_1}{a_1+1}$	$\frac{p_2}{a_2+1}$	$\frac{p_3}{a_3+1}$	a_1	a_2	a_3	q_1	q_2	q_3	a_1	a_2	a_3
0				0	0	0						
1	7	22	**71**	0	0	1	0.07	0.22	0.71	0	0	1
2	7	22	**35.5**	0	0	2	0.14	0.44	1.42	0	0	2
3	7	22	**23.67**	0	0	3	0.21	0.66	2.13	0	0	3
4	7	**22**	~~17.75~~	0	1	3	0.28	0.88	.84	0	1	3
5	7	11	**17.75**	0	1	4	0.35	1.1	3.55	0	1	4
6	7	11	**14.2**	0	1	5	0.42	1.32	4.26	0	1	5
7	7	**11**	~~11.83~~	0	1	6	0.49	1.54	~~4.97~~	0	2	5
8	7	7.33	**11.83**	0	2	6	0.56	1.76	5.68	0	2	6
9	7	7.33	**10.14**	0	2	7	0.63	1.98	6.39	0	2	7
10	7	7.33	**8.88**	0	2	8	0.7	2.2	7.1	0	2	8
11	7	**7.33**	~~7.89~~	0	2	9	0.77	2.42	~~7.81~~	0	3	8
12	7	5.5	**7.89**	0	3	9	0.84	2.64	8.52	0	3	9
13	7	5.5	**7.1**	0	3	10	0.91	2.86	9.23	0	3	10