

# Expected Value and Mixed Strategies

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## Definition

- Given a sample space with  $n$  outcomes
- With probability distribution  $P = (p_1, p_2, \dots, p_n)$ .
- Let  $X$  be a random variable that assigns the payoff  $x_k$  to outcome  $k$ .
- We define the **expected value** of  $X$  to be

$$E = E(X) = p_1x_1 + p_2x_2 + \dots + p_nx_n.$$

- Idea: average payoff per play if you play a bunch of times.
- If you run your process a hundred times, you will get about  $100 \cdot E(X)$ .

## Definition

The **expected value principle** says a rational player will choose the option with the largest expected value.

- Can be seen as a definition of “rational”
- Often taken as prescriptive: a player *should* choose
- Has many caveats! Just one model of decision-making.

## Definition

- Consider an  $m \times n$  zero-sum two-player game
- The strategies corresponding to rows and columns are now called **pure strategies**
- A **mixed strategy** for Row is a probability distribution  $P = (p_1, \dots, p_m)$  on their set of pure strategies
  - Row chooses a pure strategy at random, choosing row  $k$  with probability  $p_k$
- A mixed strategy for Column is a probability distribution  $Q = (q_1, \dots, q_n)$  on their set of pure strategies.

# Mixed Strategies

- Take  $Q = (1/4, 1/2, 1/4)$

	Rock	Paper	Scissors
Rock	0	-1	1
Paper	1	0	-1
Scissors	-1	1	0

1/4      1/2      1/4

- If Row plays Rock:  $E = \frac{1}{4} \cdot (0) + \frac{1}{2} \cdot (-1) + \frac{1}{4} \cdot (1) = -\frac{1}{4}$ 
  - On average, Row will lose one point per four games.
- If Row plays Paper:  $E = \frac{1}{4} \cdot (1) + \frac{1}{2} \cdot (0) + \frac{1}{4} \cdot (-1) = 0$
- If Row plays Scissors:  $E = \frac{1}{4} \cdot (-1) + \frac{1}{2} \cdot (1) + \frac{1}{4} \cdot (0) = \frac{1}{4}$
- What should Row do? Play Scissors.

# Mixed Strategies

## Discussion Question

Is “Play Rock every time” a mixed strategy?

## Definition

- A **basic mixed strategy** is a probability distribution with every probability except one equal to 0.
  - Write  $P_i$  for Row's basic mixed strategy that sets  $p_i = 1$  and always plays row  $i$
  - Write  $Q_j$  for Column's basic mixed strategy that sets  $q_j = 1$  and always plays column  $j$ .
- 
- We'll start by studying what happens when one player uses a pure strategy against the other player's mixed strategy
  - Later we'll generalize.

# Mixed Strategies and Expected Value

## Lemma

Consider a  $m \times n$  matrix game.

- If Row plays  $P_i$  against Column's mixed  $Q = (q_1, \dots, q_n)$ , then the expected value of the payoff is

$$E(P_i, Q) = q_1 u_{i,1} + q_2 u_{i,2} + \dots + q_n u_{i,n}.$$

- Similarly, if Column plays  $Q_j$  against Row's mixed  $P = (p_1, \dots, p_m)$ , then the expected value of the payoff is

$$E(P, Q_j) = p_1 u_{1,j} + p_2 u_{2,j} + \dots + p_m u_{m,j}.$$

## Corollary

If  $P_i$  and  $Q_j$  are basic mixed strategies, then  $E(P_i, Q_j) = u_{i,j}$ .

# Independent Probabilities

- Often want to think about two separate random processes
- E.g. Row chooses a strategy and Column chooses a strategy
- Sometimes their choices depend on each other, but usually not

## Definition

- Suppose we have two random processes:
  - $P = (p_1, \dots, p_m)$
  - $Q = (q_1, \dots, q_n)$
- The processes are **independent** if the probability of the compound outcome  $(i, j)$  is  $p_i q_j$ .

## Example (Two coins)

- Flip two (fair) coins
  - Each has sample space  $\{H, T\}$
  - Each has  $P = (1/2, 1/2)$
- Flip both coins:
  - Sample space is  $\{HH, HT, TH, TT\}$
  - Each has probability  $\frac{1}{2} \cdot \frac{1}{2} = \frac{1}{4}$
  - $P = (1/4, 1/4, 1/4, 1/4)$ .

## Example

- Tape two coins together head-to-tail
  - Each has sample space  $\{H, T\}$
  - Each has  $P = (1/2, 1/2)$
- But overall sample space is  $\{HH, TT\}$
- Probability *both* coins land heads-up is  $1/2$
- Overall probability is  $(1/2, 1/2)$ .
- Not independent.

# Independent Probabilities

## Example

- Class year and age?
  - Not independent
- Height and last digit of birthday?
  - Independent
- Political party and willingness to answer political polls?
  - Not independent unfortunately

# Independent Probabilities

- Lots of things we want to measure are *correlated*, and thus not independent
- Independence is a major assumption underlying a lot of statistical tools
- A lot of statistical work goes into dealing with that problem
- When you read a study, think about independence
- But when I give two distributions in a class exercise, assume they're independent.
- In particular, Row's mixed strategy and Column's mixed strategy are independent
  - Unless Column can read Row's mind.

# Independent Mixed Strategies

## Example (Mixed Strategies in Rock Paper Scissors)

- Row plays  $P = (1/4, 1/2, 1/4)$
- Column plays  $Q = (1/6, 1/3, 1/2)$

	R	P	S	
R	0	-1	1	1/4
P	1	0	-1	1/2
S	-1	1	0	1/4
	1/6	1/3	1/2	

	R	P	S	
R	1/24	1/12	1/8	1/4
P	1/12	1/6	1/4	1/2
S	1/24	1/12	1/8	1/4
	1/6	1/3	1/2	

# Independent Mixed Strategies

## Example (Mixed Strategies in Rock Paper Scissors)

	R	P	S	
R	0	-1	1	1/4
P	1	0	-1	1/2
S	-1	1	0	1/4
	1/6	1/3	1/2	

	R	P	S	
R	1/24	1/12	1/8	1/4
P	1/12	1/6	1/4	1/2
S	1/24	1/12	1/8	1/4
	1/6	1/3	1/2	

$$\begin{aligned} E(P, Q) &= \frac{1}{24} \cdot (0) + \frac{1}{12} \cdot (-1) + \frac{1}{8} \cdot (1) \\ &\quad + \frac{1}{12} \cdot (1) + \frac{1}{6} \cdot (0) + \frac{1}{4} \cdot (-1) \\ &\quad + \frac{1}{24} \cdot (-1) + \frac{1}{12} \cdot (1) + \frac{1}{8} \cdot (0) = -1/12. \end{aligned}$$

# Independent Mixed Strategies

## Lemma

- $m \times n$  game with payoffs  $u_{i,j}$
- Row plays  $P = (p_1, \dots, p_m)$
- Column plays  $Q = (q_1, \dots, q_n)$
- Then the probability of the outcome  $(i, j)$  is  $p_i q_j$ ,
- Expected value of the payoff is the sum of the numbers  $p_i \cdot q_j \cdot u_{i,j}$  for all values of  $i$  and  $j$ .

$$\begin{aligned} E(P, Q) &= p_1 \cdot q_1 \cdot u_{1,1} + p_2 \cdot q_1 \cdot u_{2,1} + \dots + p_m \cdot q_1 \cdot u_{m,1} \\ &\quad + p_1 \cdot q_2 \cdot u_{1,2} + p_2 \cdot q_2 \cdot u_{2,2} + \dots + p_m \cdot q_2 \cdot u_{m,2} \\ &\quad \vdots \\ &\quad + p_1 \cdot q_n \cdot u_{1,n} + p_2 \cdot q_n \cdot u_{2,n} + \dots + p_m \cdot q_n \cdot u_{m,n} \end{aligned}$$

# Independent Mixed Strategies

## Lemma

- $m \times n$  game with payoffs  $u_{i,j}$
- Row plays  $P = (p_1, \dots, p_m)$
- Column plays  $Q = (q_1, \dots, q_n)$

Then the expected value of the payoff is

$$E(P, Q) = p_1 E(P_1, Q) + \dots + p_m E(P_m, Q)$$

$$E(P, Q) = q_1 E(P, Q_1) + \dots + q_n E(P, Q_n).$$

## Proof.

Expand out either of these sums, to see that either one adds up all the  $p_i q_j u_{i,j}$ . □

# Optimal Responses to Mixed Strategies

- Some games have saddle points
- In that case, both players will want to play the saddle point
- Game will converge to a stable equilibrium
- If there's no saddle point, no pure strategy gives a stable equilibrium
- John von Neumann proved every two-person zero sum game has an optimal, equilibrium mixed strategy.
- First step: find the optimal response to a given mixed strategy.

# Optimal Responses to Mixed Strategies

## Example

- In Rock Paper Scissors, Column plays  $Q = (2/7, 4/7, 1/7)$

	Rock	Paper	Scissors
Rock	0	-1	1
Paper	1	0	-1
Scissors	-1	1	0

$2/7$        $4/7$        $1/7$

- What should Row do?
- What do you guess before computing anything?
- Let's compute expected value of each response.

# Optimal Responses to Mixed Strategies

## Example

	Rock	Paper	Scissors
Rock	0	-1	1
Paper	1	0	-1
Scissors	-1	1	0
	2/7	4/7	1/7

$$E(P_1, Q) = \frac{2}{7} \cdot (0) + \frac{4}{7} \cdot (-1) + \frac{1}{7} \cdot (1) = -3/7$$

$$E(P_2, Q) = \frac{2}{7} \cdot (1) + \frac{4}{7} \cdot (0) + \frac{1}{7} \cdot (-1) = 1/7$$

$$E(P_3, Q) = \frac{2}{7} \cdot (-1) + \frac{4}{7} \cdot (1) + \frac{1}{7} \cdot (0) = 2/7$$

- Best pure strategy: play Scissors

# Optimal Responses to Mixed Strategies

- Can find the best pure strategy response

## Discussion Question

Can we do better with a mixed strategy?