Neutralizing Strategies and Nash Equilibria

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Independent Mixed Stratgies

Lemma

- *$m \times n$ game with payoffs $u_{i,j}$*
- *Row plays $P = (p_1, \ldots, p_m)$*
- *Column plays $Q = (q_1, \ldots, q_n)$*
- *Then the probability of the outcome (i, j) is $p_i q_j$,*
- *Expected value of the payoff is the sum of the numbers $p_i \cdot q_j \cdot u_{i,j}$ for all values of i and j.*

$$\begin{aligned}
E(P, Q) = {} & p_1 \cdot q_1 \cdot u_{1,1} + p_2 \cdot q_1 \cdot u_{2,1} + \cdots + p_m \cdot q_1 \cdot u_{m,1} \\
& + p_1 \cdot q_2 \cdot u_{1,2} + p_2 \cdot q_2 \cdot u_{2,2} + \cdots + p_m \cdot q_2 \cdot u_{m,2} \\
& \vdots \\
& + p_1 \cdot q_n \cdot u_{1,n} + p_2 \cdot q_n \cdot u_{2,n} + \cdots + p_m \cdot q_n \cdot u_{m,n}
\end{aligned}$$

Independent Mixed Stratgies

Lemma

- *$m \times n$ game with payoffs $u_{i,j}$*
- *Row plays $P = (p_1, \ldots, p_m)$*
- *Column plays $Q = (q_1, \ldots, q_n)$*

Then the expected value of the payoff is

$$E(P,Q) = p_1 E(P_1, Q) + \cdots + p_m E(P_m, Q)$$
$$E(P,Q) = q_1 E(P, Q_1) + \cdots + q_n E(P, Q_n).$$

Proof.

Expand out either of these sums, to see that either one adds up all the $p_i q_j u_{i,j}$. □

Optimal Responses to Mixed Strategies

- Some games have saddle points
- In that case, both players will want to play the saddle point
- Game will converge to a stable equilibrium
- If there's no saddle point, no pure strategy gives a stable equilibrium
- John von Neumann proved every two-person zero sum game has an optimal, equilibrium mixed strategy.
- First step: find the optimal response to a given mixed strategy.

Optimal Responses to Mixed Strategies

Example

	Rock	Paper	Scissors
Rock	0	-1	1
Paper	1	0	-1
Scissors	-1	1	0
	2/7	4/7	1/7

$$E(P_1, Q) = \frac{2}{7} \cdot (0) + \frac{4}{7} \cdot (-1) + \frac{1}{7} \cdot (1) = -3/7$$

$$E(P_2, Q) = \frac{2}{7} \cdot (1) + \frac{4}{7} \cdot (0) + \frac{1}{7} \cdot (-1) = 1/7$$

$$E(P_3, Q) = \frac{2}{7} \cdot (-1) + \frac{4}{7} \cdot (1) + \frac{1}{7} \cdot (0) = 2/7$$

- Best pure strategy: play Scissors

Optimal Responses to Mixed Strategies

- Can find the best pure strategy response
- Can we do better with a mixed strategy?

Lemma

There is always a pure strategy among the best responses a player has to any pure or mixed strategy played by their opponent.

Proof.

Idea:

- A mixed strategy is a combination or average of pure strategies
- The average can't be better than the best component of that average

Optimal Responses to Mixed Strategies

Lemma

There is always a pure strategy among the best responses a player has to any pure or mixed strategy played by their opponent.

Proof.

- Suppose P_k is Row's best response to Column's mixed Q.
- That means $E(P_k, Q) \geq E(P_i, Q)$ for any row i
- Let P be a mixed strategy. Then

$$\begin{aligned} E(P, Q) &= p_1 E(P_1, Q) + p_2 E(P_2, Q) + \cdots + p_m E(P_m, Q) \\ &\leq p_1 E(P_k, Q) + p_2 E(P_k, Q) + \cdots + p_m E(P_k, Q) \\ &= (p_1 + p_2 + \cdots + p_m) E(P_k, Q) = E(P_k, Q). \end{aligned}$$

□

Optimal Responses to Mixed Strategies

Lemma

There is always a pure strategy among the best responses a player has to any pure or mixed strategy played by their opponent.

- Often counterintuitive
- If opponent plays rock half the time, should you play paper half the time?
- This theorem says no. All paper all the time.

Discussion Question

Why would we ever use mixed strategies?

- Mixed strategies don't help exploit your opponent
- They can stop your opponent from exploiting you.

Nash Equilibria

Definition

A mixed strategy outcome (P, Q) is an equilibrium or Nash equilibrium if

- P is a best response to Q, and
- Q is a best response to P.

We call P and Q equilibrium strategies.

- Named after mathematician John Nash
- Seen in *A Beautiful Mind* (2001 Oscar-winning film)
- (P, Q) is an equilibrium if and only if:

$$E(P, Q) \geq E(R, Q) \qquad \text{for any Row mixed strategy } R$$

$$E(P, Q) \leq E(P, S) \qquad \text{for any Column mixed strategy } S$$

Nash Equilibria

- Nash equilibria generalize saddle points to mixed strategies.

Lemma

A pure strategy outcome (k, ℓ) *is a saddle point if and only if the corresponding basic mixed strategy outcome* (P_k, Q_ℓ) *is a Nash equilibrium.*

Nash Equilibria

Example (Rock Paper Scissors)

- $Q = (2/7, 4/7, 1/7)$ isn't equilibrium
- Row can exploit by playing all scissors, with $P = P_3 = (0, 0, 1)$.
- What about $Q = (1/3, 1/3, 1/3)$?

$$E(P_1, Q') = \frac{1}{3} \cdot (0) + \frac{1}{3} \cdot (-1) + \frac{1}{3} \cdot (1) = 0$$
$$E(P_2, Q') = \frac{1}{3} \cdot (1) + \frac{1}{3} \cdot (0) + \frac{1}{3} \cdot (-1) = 0$$
$$E(P_3, Q') = \frac{1}{3} \cdot (-1) + \frac{1}{3} \cdot (1) + \frac{1}{3} \cdot (0) = 0$$

- Row's expected payoff is 0, regardless of strategy.

Nash Equilibria

Definition

- A mixed strategy is called a neutralizing strategy if the expected payoff is the same for every possible response by the opponent.
- An outcome (P, Q) is a neutralizing outcome if both P and Q are neutralizing strategies.

Lemma

A neutralizing outcome in a zero-sum game is a Nash equilibrium.

Nash Equilibria

Lemma

A neutralizing outcome in a zero-sum game is a Nash equilibrium.

Proof.

- Let (P, Q) be a neutralizing outcome.
- Then P is a neutralizing strategy for Row
  - So *every* Column strategy is a best response to P
  - In particular, Q is a best response to P.
- Similarly, P is a best response to Q.
- So this is a Nash equilibrium.

□

Nash Equilibria

Theorem (von Neumann's equilibrium theorem)

Every two-person zero-sum game has a Nash equilibrium.

Proof.

- Full proof is tricky
- We'll prove a limited version for 2×2 games.

Definition

The equilibrium method for a zero-sum game is the method in which players choose one of their equilibrium strategies.

Nash Equilibria

Example (Rock Paper Scissors)

- The neutralizing strategy for Row is $P = (1/3, 1/3, 1/3)$
- The neutralizing strategy for Column is $Q = (1/3, 1/3, 1/3)$
- That is, the neutralizing strategy chooses from the three options with equal probability.
- By our theorem, this is a Nash equilibrium.

Discussion Question

How do we find these?

Prudent Mixed Strategies

Definition

The guarantee of a mixed strategy is the expected value of the payoff when the opponent plays their best response.

- Check the expected value of every pure response.

Example

- Consider $Q = (2/7, 4/7, 1/7)$
- Computed $E(R) = -3/7$, $E(P) = 1/7$, $E(S) = 2/7$
- The guarantee of Q is $2/7$
- Now consider $Q' = (1/3, 1/3, 1/3)$
- Saw $E(R) = E(P) = E(S) = 0$
- The guarantee of Q' is 0.

Prudent Mixed Strategies

Definition

- The prudent mixed strategy for a player is the mixed strategy with the best guarantee.
  - (Highest guarantee for Row, and lowest for Column)
  - We write $\overline{r}$ and $\overline{c}$ for these best guarantees
- The prudent mixed strategy method is the method in which each player plays their prudent mixed strategy.

Claim

Every game has a prudent mixed strategy.

Prudent Mixed Strategies

Example (Rock Paper Scissors)

- $P' = (1/3, 1/3, 1/3)$ is Row's unique prudent mixed strategy
- $Q' = (1/3, 1/3, 1/3)$ is Column's unique prudent mixed strategy
- Guarantees are $\overline{r} = 0$ and $\overline{c} = 0$.

Proof.

- Suppose $P = (p_1, p_2, p_3)$ is some other strategy
- The biggest p_i must be bigger than $1/3$
- If Column plays the counterplay to the most probable strategy, they will win on average
- So the guarantee of P is negative, which is less than $\overline{r} = 0$.

□

von Neumann's Min-Max Theorem

Theorem (von Neumann's Min-Max Theorem)

In a two-person zero-sum game:

- *Every Nash equilibrium (P, Q) is doubly prudent*
- $\overline{r} = \overline{c}$
- *There are always prudent mixed strategies for both players*
- *Every doubly prudent mixed strategy is a Nash equilibrium.*

von Neumann's Min-Max Theorem

Claim

If (P, Q) is a Nash equilibrium then it's doubly prudent

Proof.

- P is a best response to Q
  - So $E(P, Q) \geq E(R, Q)$ for any other mixed strategy R
  - Thus $E(P, Q)$ is the guarantee of Column's Q
- Similarly, Q is a best response to P
  - So $E(P, Q) \leq E(P, S)$ for any other S
  - Thus $E(P, Q)$ is the guarantee of Row's P
- $E(P, Q) \leq \overline{r}$ and $\overline{c} \leq E(P, Q)$
- But $\overline{r} \leq \overline{c}$ so $E(P, Q)$ is both players' best guarantee.

von Neumann's Min-Max Theorem

Claim

In any two-player zero-sum game, $\overline{r} = \overline{c}$.

Proof.

- By von Neumann's equilibrium theorem, there is a Nash equilibrium
- Just saw any Nash equilibrium has $\overline{c} \leq E(P, Q) \leq \overline{r}$
- But $\overline{r} \leq \overline{c}$
- Thus $E(P, Q) = \overline{r} = \overline{c}$.

von Neumann's Min-Max Theorem

Claim

If (P, Q) is doubly prudent then it's a Nash equilibrium.

Proof.

- Since P is prudent, $E(P, S) \geq \overline{r}$ for any strategy S.
- Similarly, since Q is prudent, $E(R, Q) \leq \overline{c}$ for any R.
- In particular, $\overline{r} \leq E(P, Q) \leq \overline{c}$
- But $\overline{r} = \overline{c}$, so $E(P, Q) = \overline{r} = \overline{c}$.
- $E(P, S) \geq E(P, Q)$ for any S, so Q is a best response to P.
- $E(R, Q) \leq E(P, Q)$ for any R, so P is a best response to Q.
- Thus (P, Q) is a Nash equilibrium.

□

Solving Games

Definition

Let (P, Q) be a Nash equilibrium for a two-player zero-sum game.

- We call the expected value $E(P, Q)$ the value v of the game.
- A solution to the game is an equilibrium strategy (P, Q) together with the value v.
- If $v = 0$ we say the game is fair.

- If a game has a saddle point, that's a solution.
- There's an algorithm for solving any 2P zero-sum game
- It's more complicated than we want to work out here.
- We will work out the 2×2 case, though
- Easier to *check* whether a given pair of strategies is a solution.

Solving Games

Example

2	-1	-1
-1	2	-1
0	-3	2

- $P = (1/5, 1/2, 3/10)$
- $Q = (3/10, 3/10, 2/5)$
- Is this a solution?

- Compute EV of each of Column's strategies against P

$$E(P, Q_1) = \frac{1}{5} \cdot (2) + \frac{1}{2} \cdot (-1) + \frac{3}{10} \cdot (0) = -1/10$$
$$E(P, Q_2) = \frac{1}{5} \cdot (-1) + \frac{1}{2} \cdot (2) + \frac{3}{10} \cdot (-3) = -1/10$$
$$E(P, Q_3) = \frac{1}{5} \cdot (-1) + \frac{1}{2} \cdot (-1) + \frac{3}{10} \cdot (2) = -1/10.$$

- Thus P is a neutralizing strategy against Column with value $v = -1/10$.

Example

2	-1	-1
-1	2	-1
0	-3	2

- $P = (1/5, 1/2, 3/10)$
- $Q = (3/10, 3/10, 2/5)$
- Is this a solution?

- Now compute EV of each of Row's strategies against Q

$$E(P_1, Q) = \frac{3}{10} \cdot (2) + \frac{3}{10} \cdot (-1) + \frac{2}{5} \cdot (-1) = -1/10$$

$$E(P_1, Q) = \frac{3}{10} \cdot (-1) + \frac{3}{10} \cdot (2) + \frac{2}{5} \cdot (-1) = -1/10$$

$$E(P_1, Q) = \frac{3}{10} \cdot (0) + \frac{3}{10} \cdot (-3) + \frac{2}{5} \cdot (2) = -1/10.$$

- Thus Q is a neutralizing strategy against Row with value $v = -1/10$.