Solving 2×2 Games

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Nash Equilibria

Definition

A mixed strategy outcome (P, Q) is an equilibrium or Nash equilibrium if

- P is a best response to Q, and
- Q is a best response to P.

We call P and Q equilibrium strategies.

Definition

- A mixed strategy is called a neutralizing strategy if the expected payoff is the same for every possible response by the opponent.
- An outcome (P, Q) is a neutralizing outcome if both P and Q are neutralizing strategies.

Solving 2×2 games

Theorem

Any 2×2 zero-sum game has a Nash equilbrium, and there's a formula to find it.

Proof.

Consider the most general possible 2×2 game:

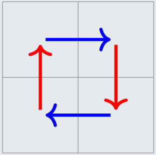
	1	2
1	a	b
2	c	d

- If there's a saddle point, that gives a Nash Equilibrium.
- Assume there's no saddle point. What happens?

Solving 2×2 games

Proof.

	1	2
1	a	b
2	c	d

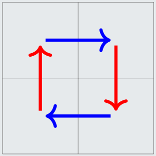


- Assume without loss of generality that $a \geq c$
  - (If $c \geq a$ swap the columns around)
- That gives a flow diagram with the left arrow pointing up
- Since there's no saddle point, we must get this flow diagram.
  - Can't have $a = c$, so $a > c$
  - Also $d > b$
  - $a > b$, $d > c$.

Solving 2×2 games

Proof.

	1	2
1	a	b
2	c	d



- Suppose Column adopts strategy $Q = (1-q, q)$. Then
$$E(P_1, Q) = a(1-q) + bq = (b-a)q + a$$
$$E(P_2, Q) = c(1-q) + dq = (d-c)q + c.$$
- These are equations of lines
- $b < a$, so first line slopes down
- $d > c$, so second line slopes up
- They intersect somewhere; find the intersection algebraically.

Solving 2×2 games

Proof.

$$\begin{aligned}(b-a)q+a &= (d-c)q+c \\ a-c &= aq-bq+dq-cq \\ &= \big((a-c)+(d-b)\big)q \\ q &= \frac{a-c}{(a-c)+(d-b)}.\end{aligned}$$

$$\begin{aligned}1-q &= 1-\frac{a-c}{(a-c)+(d-b)} \\ &= \frac{(a-c)+(d-b)}{(a-c)+(d-b)} - \frac{a-c}{(a-c)+(d-b)} \\ &= \frac{d-b}{(a-c)+(d-b)}.\end{aligned}$$

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Proof.

- Column's strategy $Q = (1-q, q)$ is neutralizing when

$$q = \frac{a-c}{(a-c)+(d-b)} \qquad 1-q = \frac{d-b}{(a-c)+(d-b)}$$

$$\begin{aligned} E(P_1, Q) &= a(1-q) + bq \\ &= a\frac{d-b}{(a-c)+(d-b)} + b\frac{a-c}{(a-c)+(d-b)} \\ &= \frac{ad-ab+ab-bc}{(a-c)+(d-b)} = \frac{ad-bc}{(a-c)+(d-b)} \\ E(P_2, Q) &= c(1-q) + dq \\ &= \ldots = \frac{ad-bc}{(a-c)+(d-b)}. \end{aligned}$$

Solving 2×2 games

Proof.

- Set $q = \dfrac{a-c}{(a-c)+(d-b)}$
- The strategy $Q = (1-q, q)$ is neutralizing
- Both of Row's pure strategies have expected value $v = \dfrac{ad-bc}{(a-c)+(d-b)}$.
- We can make the same series of arguments for Row.
- Set $p = \dfrac{a-b}{(a-b)+(d-c)}$
- Then the strategy $R = (1-p, p)$ is neutralizing
- $E(P, Q_1) = E(P, Q_2) = \dfrac{ad-bc}{(a-b)+(d-c)}$.

□

Solving 2×2 games

Corollary

- *Any 2×2 zero-sum game has a Nash equilbrium.*
- *If there's no saddle point, we have a formula for the mixed strategy Nash equilibrium.*
- *Set* $p = \dfrac{a-b}{(a-b)+(d-c)}$ *and* $q = \dfrac{a-c}{(a-c)+(d-b)}$
- *Then Row's equilibrium strategy is* $P = (1-p, p)$
- *Column's equilibrium strategy is* $Q = (1-q, q)$
- *The value of the game is* $v = \dfrac{ad-bc}{(a-b)+(d-c)}$.

Solving 2×2 games

Remark

The number $ad - bc$ is sometimes called the determinant of the matrix

a	b
c	d

. It is frequently important in math contexts. It is sometimes notated Δ.

- Want to show how this sort of solution works in practice
- Also want to show why we should care.
- Rarely wind up having to play a 2×2 matrix game in real life
- But many situations are *like* 2×2 matrix games.

An Application to Counterterrorism

Example (The Hunt for Osama bin Laden)

- Osama bin Laden organized the September 11, 2001 terror attack on the World Trade Center
- The US occupied Afghanistan pursuing him
- December 2002: hiding on the Afghanistan-Pakistan border
- The US could search in Afghanistan or in Pakistan
- Bin Laden could hide in either Afghanistan or Pakistan
- Need to estimate likelihood of catching him

An Application to Counterterrorism

Example (The Hunt for Osama bin Laden)

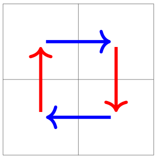
- If US searches Afghanistan and he's there: 60% chance
- If US searches Pakistan and he's there: 40% chance
- If US searches Pakistan and he's in Afghanistan: 0% chance
- If US searches Afghanistan and he's in Pakistan: 10% chance
  Pakistani security catches him

	A	P
A	60	10
P	0	40

- What should the US do? What should bin Laden do?

An Application to Counterterrorism

	A	P
A	60	10
P	0	40



- Is there a saddle point? No
- Use our formula to find an equilibrium mixed strategy

$$p = \frac{a-b}{(a-b)+(d-c)} = \frac{60-10}{60-10+40-0} = \frac{50}{90} = \frac{5}{9}$$

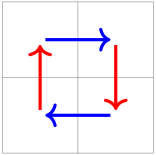
$$1-p = \frac{9}{9} - \frac{5}{9} = \frac{4}{9}$$

$$q = \frac{a-c}{(a-c)+(d-b)} = \frac{60-0}{60-0+40-10} = \frac{60}{90} = \frac{2}{3}$$

$$1-q = 1 - \frac{2}{3} = \frac{1}{3}$$

An Application to Counterterrorism

	A	P
A	60	10
P	0	40



- US should play A 4/9 of the time and P 5/9 of the time.
- ObL should play A 1/3 of the time and P 2/3 of the time.
- Value of the game is

$$v = \frac{ad - bc}{(a-b)+(d-c)} = \frac{(60)(40)-(10)(0)}{60-10+40-0}$$
$$= \frac{2400-0}{90} = \frac{240}{9} = \frac{80}{3} = 26^2/_3.$$

An Application to Counterterrorism

- $P = (4/9, 5/9)$ $\qquad Q = (1/3, 2/3)$ $\qquad v = 26{}^2\!/\!{}_3$
- How do we interpret this?
- One option: US randomly chooses whether to search in Afghanistan or in Pakistan, with non-uniform probabilities
- Another option: send $4/9$ of resources to Afghanistan and $5/9$ of resources to Pakistan.
- In any event: all these numbers are estimates. Don't take "$26{}^2\!/\!{}_3$%" too seriously.
- But this can guide how we think about these sorts of decisions.

Example: Battle of the Bismarck Sea

	North	South
North	2	2
South	1	3

- $p = \dfrac{a-b}{(a-b)+(d-c)}$
- $q = \dfrac{a-c}{(a-c)+(d-b)}$
- $v = \dfrac{ad-bc}{(a-b)+(d-c)}$.

- $p = \dfrac{2-2}{(2-2)+(3-1)} = 0$
- $q = \dfrac{2-1}{(2-1)+(3-2)} = 1/2$
- $v = \dfrac{2 \cdot 3 - 1 \cdot 2}{(2-2)+(3-1)} = 4/2 = 2.$
- Row plays North all the time; Column plays 50-50. The expected payout is 2.
- How does this compare to our original answers?

Another example

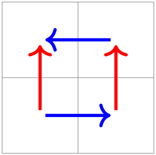
-1	1
-3	-4

- $p = \dfrac{a-b}{(a-b)+(d-c)}$
- $q = \dfrac{a-c}{(a-c)+(d-b)}$
- $v = \dfrac{ad-bc}{(a-b)+(d-c)}$.

- $p = \dfrac{-1-1}{(-1-1)+(-4+3)} = \dfrac{-2}{-3} = \dfrac{2}{3}$
- $q = \dfrac{-1+3}{(-1+3)+(-4-1)} = \dfrac{2}{-3} = -\dfrac{2}{3}$
- $v = \dfrac{(-1)(-4)-(-3)(1)}{(-1-1)+(-4+3)} = \dfrac{7}{-3} = -\dfrac{7}{3}$.
- How can a probability be negative?
- Something went wrong.

Another example

-1	1
-3	-4



- Saddle point at (1,1)
- Our mixed strategy formula assumed no saddle point
- Always check first!

Escaping the conflict

- This is a framework for conflict with another player
- Zero sum: no room for cooperation
- Can win or lose a war, but can't avoid it
- Can we model the possibility of cooperation?