

Criteria for Voting Systems with Multiple Candidates

Jay Daigle

`jaydaigle@gwu.edu`

`https://jaydaigle.net/politics`

The George Washington University

January 27, 2026

Multi-Candidate Voting Systems

- Plurality
- Hare's method (Instant Runoff Voting)
- Coombs's Method
- Borda Count
- Copeland's Method

Discussion Question

- How do we decide which of these are good?
- What do we want out of a multi-candidate election?

Some Bad Options

Definition

In the **dictatorship method**, one voter is the **dictator**. Their first-choice candidate is the unique winner.

Definition

In the **monarchy method**, one candidate is the **monarch**. That candidate is the unique winner regardless of how anyone votes.

Definition

In the **all-ties method**, every candidate is selected as a winner.

Discussion Question

- When might each of these be a good idea?
- Why is each of these usually a bad idea?

Some Easy Criteria

Definition

A method satisfies the **Unanimity criterion**, or is **unanimous**, if, whenever all voters place the same candidate at the top of their preference orders, that candidate is the unique winner.

- Obviously desirable, but too easy.

Definition

A method is **decisive** if it always selects a unique winner.

- Obviously desirable, but too hard!

Definition

A method satisfies the **majority criterion** if, whenever a candidate receives a majority of the first-place votes, that candidate must be the unique winner.

Definition

A method is **anonymous** if the outcome is unchanged whenever two voters exchange their ballots.

Lemma

A social choice function is anonymous if and only if it depends only on the tabulated profile.

Proof.

The same as the proof in the two-candidate case.

Definition

A method is **neutral** if it treats all candidates the same:

- Suppose we have some profile that names A to be a winner.
 - Now suppose there is another candidate B , and all voters exactly swap their preferences for A and B .
 - In the new profile, B should be a winner.
-
- Almost every method we consider is anonymous and neutral.
 - Important to name and articulate, especially for proofs.
 - Which methods aren't anonymous?
 - Which methods aren't neutral?

Monotonicity

- Idea: gaining more support shouldn't hurt you.

Definition

A method is **monotone** if:

- Suppose there is a profile in which
 - Candidate A wins
 - But some voter puts another candidate B immediately ahead of A .
- If that voter moves A up one place to be ahead of B ,
- Then A must be a winner in the new profile.

Corollary

In a monotone method, if A moves up any number of places on any number of ballots, they should still win.

Discussion Question

A	C	B
B	A	C
C	D	D
D	B	A

Who *shouldn't* win?

Definition

A method is **Pareto** or satisfies the **Pareto criterion** if whenever every voter prefers a candidate A to another candidate B , then the method does not select B as a winner.

Definition

A method is **Pareto** or satisfies the **Pareto criterion** if whenever every voter prefers a candidate A to another candidate B , then the method does not select B as a winner.

- Named after Italian economist Vilfredo Pareto (1848-1923)
- In general: a choice is **Pareto optimal** if no other choice is better for everyone.
- Pareto criterion guarantees winner is Pareto optimal.
- Does *not* guarantee A wins!

Definition

A method is **Pareto** or satisfies the **Pareto criterion** if whenever every voter prefers a candidate A to another candidate B , then the method does not select B as a winner.

Example

7	6
D	C
C	D
A	A
B	B

- Does not guarantee A wins!
- All prefer A to B , so B can't win
- But all prefer C to A , so A can't win either
- Do we know who does win?

Definition

- A candidate is a **Condorcet candidate** if they beat every other candidate in a head-to-head (by simple majority).
 - They are an **anti-Condorcet candidate** if they lose to every other candidate in a head-to-head.
-
- Marie Jean Antoine Nicolas de Caritat, Marquis of Condorcet (1743-1794)
 - Nicolas de Condorcet or Marquis de Concorcet
 - One of the first theorists of voting
 - *Essay on the Application of Analysis to the Probability of Majority Decisions* (1785)

Definition

- A candidate is a **Condorcet candidate** if they beat every other candidate in a head-to-head (by simple majority).
- They are an **anti-Condorcet candidate** if they lose to every other candidate in a head-to-head.

Example

A	B	C
B	C	A
C	A	B

- A beats B
- B beats C
- C beats A
- No Condorcet candidate
- No anti-Condorcet candidate

Definition

- A candidate is a **Condorcet candidate** if they beat every other candidate in a head-to-head (by simple majority).
- They are an **anti-Condorcet candidate** if they lose to every other candidate in a head-to-head.

Definition

- A method satisfies the **Condorcet criterion** if whenever there's a Condorcet candidate, they're the unique winner.
- A method satisfies the **anti-Condorcet criterion** if whenever there's an anti-Condorcet candidate, they don't win.

Definition

- A method is **independent** if it satisfies the following (somewhat complicated) property:
- Suppose there are two profiles where no voter changes their mind about whether candidate A is preferred to candidate B : if a voter ranks A above B in the first profile, they also rank A above B in the second profile
- If A wins in the first profile but B doesn't, then B cannot win in the second profile.
- (This also works backwards: if B wins in the second profile, they can't lose in the first.)

Independence

- Kenneth Arrow (1950)
- “Independence of irrelevant alternatives” or “IIA”

After finishing dinner, [Columbia philosopher] Sidney Morgenbesser decides to order dessert. The waitress tells him he has two choices: apple pie and blueberry pie. Sidney orders the apple pie. After a few minutes the waitress returns and says that they also have cherry pie at which point Morgenbesser says “In that case I’ll have the blueberry pie.”

Independence

Example

Profile 1						
A	C	B	C	C	A	B
B	A	C	B	B	B	A
C	B	A	A	A	C	C

Profile 2						
A	C	B	C	C	A	B
B	A	C	B	B	B	C
C	B	A	A	A	C	A

- Relative ranks of A and B haven't changed.
- If A wins and B loses in Profile 1, then B shouldn't win in Profile 2.

Independence

- Why is the phrasing so complicated?
- *Idea*: if A beats B in profile 1, A still beats B in profile 2.
- But a move could make A and B both lose.

Example

Profile 1			Profile 2		
A	A	C	A	C	C
C	C	A	C	A	A
B	B	B	B	B	B

- Change shouldn't make B win but it can make A lose
- We don't name second place, so A doesn't "beat" B.

Our First Impossibility Result

Proposition

Any social choice function that satisfies anonymity and neutrality must violate decisiveness.

Proof.

- Since our method is anonymous, can consider tabulated profiles
- Suppose we have $2n$ voters and 2 candidates
- Consider two profiles:

n	n	n	n
A	B	B	A
B	A	A	B

Our First Impossibility Result

Proposition

Any social choice function that satisfies anonymity and neutrality must violate decisiveness.

Proof.

n	n
A	B
B	A

n	n
B	A
A	B

- At least one candidate wins.
- Assume A wins in the top profile.
- By neutrality B wins in the bottom.
- Profiles are the same, so B wins in top.
- Top profile does not have a unique winner.
- Therefore the method can't be decisive.



Taylor's Theorem

Proposition (Taylor)

No social choice function involving at least three candidates satisfies both independence and the Condorcet criterion.

Proof.

- Suppose we have an independent Condorcet method.
- Consider this profile:

A	C	B
B	A	C
C	B	A

- Claim no candidate can be a winner.



Taylor's Theorem

A	C	B
B	A	C
C	B	A

A	C	C
B	A	B
C	B	A

Claim

A can't win under independence and Condorcet

Proof.

- Consider this profile 2
- By Condorcet, C must be unique winner in profile 2
- C wins and A loses in profile 2
- Only swapped C and B, so by independence A loses in profile 1.



Taylor's Theorem

Proposition (Taylor)

No social choice function involving at least three candidates satisfies both independence and the Condorcet criterion.

Proof.

A	C	B
B	A	C
C	B	A

- Assume we have an independent Condorcet method for this profile
- Showed A can't win this profile
- Same argument shows B can't win
- Same argument shows C can't win
- Our method can't name a winner for this profile.



Proposition

If a method is Condorcet then it satisfies the majority criterion.

Proof.

- Suppose A has a majority of first-place votes.
- Then A will win any head-to-head matchup, so A is the Condorcet candidate.
- Any method that satisfies the Condorcet criterion will cause A to win, also satisfying the majority criterion.



Remark

We can say the Condorcet criterion is “stronger” than the majority criterion.

Summary

Voting Methods

- Plurality
- Hare's method
- Coombs's Method
- dictatorship
- monarchy
- all ties
- Copeland's Method
- Borda count
- More to come!

Voting Method Criteria

- unanimous
- decisive
- majoritarian
- anonymous
- neutral
- monotone
- Pareto
- independent
- Condorcet
- anti-Condorcet

Analyzing the Plurality Method

Proposition

The plurality method is majoritarian, monotone, and Pareto, but not Condorcet, anti-Condorcet, or independent.

Proof.

- The majority is always a plurality, so candidate with majority will win.
- Monotone: raising a candidate on some preference lists can't reduce their first-place votes, or increase anyone else's
- Pareto: if A is ahead of B on every preference list, then B gets no votes, and can't win.

Analyzing the Plurality Method

Claim

The plurality method is neither Condorcet nor anti-Condorcet.

Proof.

2	3	2
A	B	C
C	A	A
B	C	B

- B wins plurality, but
- A beats B, A beats C, and C beats B.
- A is Condorcet, and loses
- B is anti-Condorcet, and wins



Analyzing the Plurality Method

Claim

The plurality method is not independent.

Proof.

A	A	A	A	B	B	B
B	B	C	C	A	A	A
C	C	B	B	C	C	C

 →

A	A	C	C	B	B	B
B	B	A	A	A	A	A
C	C	B	B	C	C	C

- First profile: A wins, B and C lose
- Second profile: B wins, A and C lose
- But relative position of A and B doesn't change.

