The Condorcet Paradox and Arrow's Theorem

Jay Daigle

jaydaigle@gwu.edu

https://jaydaigle.net/politics

The George Washington University

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Summary of Voting Method Properties

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Plurality	Y	Y	Y	N	Y	N	N	Y	Y	N
Antiplur	Y	Y	Y	N	N	N	N	Y	N	N
Borda	Y	Y	Y	N	N	N	Y	Y	Y	N
Hare	Y	Y	Y	N	Y	N	N	N	Y	N
Coombs	Y	Y	Y	N	N	N	N	N	Y	N
Copeland	Y	Y	Y	N	Y	Y	Y	Y	Y	N
Black	Y	Y	Y	N	Y	Y	Y	Y	Y	N
Dictator	N	Y	Y	Y	N	N	N	Y	Y	Y
All-ties	Y	Y	N	N	N	N	N	Y	N	Y
Monarchy	Y	N	N	Y	N	N	N	Y	N	Y

Tradeoffs

Proposition (Taylor)

No social choice function involving at least three candidates satisfies both independence and the Condorcet criterion.

Discussion Question

Condorcet isn't that common. Why is independence so hard?

Theorem (Arrow's Impossibility Theorem)

If a social choice function with at least three candidates satisfies both Pareto and independence, then it must be a dictatorship.

Outline of the Proof

Lemma (decisiveness lemma)

A social choice function with at least three candidates that satisfies Pareto and independence must be decisive.

Lemma

Suppose a social choice function with at least three candidates satisfies Pareto and independence. Suppose there are two profiles in which no voter changes their mind about whether candidate A is preferred to candidate B. If A wins in the first profile, then B cannot win in the second profile.

Discussion Question

How is this different from independence alone?

Outline of the Proof

- Suppose we have a social choice function with more than three candidates that is Pareto and independent.
- It must be decisive.
- It satisfies the "super independence" property.

Claim

For each pair of candidates, there is some single voter who can force one candidate to lose by ranking the other candidate higher.

Claim

This voter must be the same for each pair of candidates.

The Decisiveness Lemma

Lemma (decisiveness lemma)

A social choice function with at least three candidates that satisfies Pareto and independence must be decisive.

Proof.

- Suppose we have a profile where A and B both win
- We will build a profile in which no candidate can win
- That's not possible, so we can't have a profile like that.

The Decisiveness Lemma

Proof.

- Suppose in some profile A and B both win
- Suppose x voters prefer A to B, and y voters prefer B to A
- $x \neq 0$ and $y \neq 0$:
- If $x = 0$ then all voters prefer B to A, and by Pareto we know A can't win.

The Decisiveness Lemma

Proof.

x	y
⋮	⋮
A	B
⋮	⋮
B	A
⋮	⋮

Profile P

New profile:

x	y
A	C
C	B
B	A
⋮	⋮

Profile Q

- We claim C is the unique winner in profile Q
- A can't win, by independence:
  - If A wins in Q, then A wins and B loses
  - Relative positions unchanged from Q to P
  - Then B can't win in P, contradicting our assumption
- B can't win, by Pareto
- Neither can anyone else

The Decisiveness Lemma

Proof.

x	y
⋮	⋮
A	B
⋮	⋮
B	A
⋮	⋮

Profile P

x	y
A	C
C	B
B	A
⋮	⋮

Profile Q

x	y
A	B
B	C
C	A
⋮	⋮

Profile R

- A loses by independence:
  - C wins and A loses in Q
  - Relative positions unchanged from Q to R
- B loses by independence:
  - Relative positions unchanged from P to R
  - A wins in P but loses in R
- No one else can win, by Pareto
- No one can win!

The Decisiveness Lemma

Lemma (decisiveness lemma)

A social choice function with at least three candidates that satisfies Pareto and independence must be decisive.

Proof.

- Suppose we have a profile where A and B both win
- We built a profile in which no candidate can win
- That's not possible, so we can't have a profile like that
- That proves the lemma.

□

A Corollary to Decisiveness

Lemma (decisiveness lemma)

A social choice function with at least three candidates that satisfies Pareto and independence must be decisive.

Corollary

It is impossible for a method to satisfy Pareto, independence, anonymity, and neutrality.

Proof.

- A Pareto and independent method must be decisive
- An anonymous and neutral method cannot be decisive
- We have to lose one.

□

Another Corollary

Lemma

Suppose a social choice function with at least three candidates satisfies Pareto and independence. Suppose there are two profiles in which no voter changes their mind about whether candidate A is preferred to candidate B. If A wins in the first profile, then B cannot win in the second profile.

Proof.

- Suppose A wins in the first profile
- Method is decisive by decisiveness lemma, so B can't win in first profile
- By independence, B can't win in second profile.

□

A New Idea: Dictatorial Control

- Want to show that a method has to be dictatorship
- Define a sort of part-way dictatorship

Definition

Suppose A and B are two candidates. We say a voter has dictatorial control for A over B if, whenever that voter prefers A to B, it will always be the case that B loses.

- Not necessarily symmetric!

Dictatorial Control

Lemma

If a social choice function with at least three candidates satisfies both Pareto and independence, then for any pair of candidates A and B, there is a voter with dictatorial control for A over B.

Outline of Proof.

- Switch preferences one at a time
- One specific voter decides between A and C
- That voter must have control for A over B by repeated independence arguments

□

Dictatorial Control

Claim

Some voter has dictatorial control for A over B

Proof.

- Suppose all voters rank C first, A second, B third
- C has to win by Pareto
- One by one, switch to A then B then C
- At the end, A wins by Pareto
- Some specific voter causes the winner to switch.

Dictatorial Control

Claim

Some voter has dictatorial control for A over B.

Proof.

X	v	Y
C	C	A
A	A	B
B	B	C
⋮	⋮	⋮

X	v	Y
C	A	A
A	B	B
B	C	C
⋮	⋮	⋮

- When v switches, that causes A to in, instead of C.
- Claim that v has dictatorial control for A over B.

Dictatorial Control

Claim

The voter v has dictatorial control for A over B.

Proof.

- Imagine profile P where v ranks A over B.
- Claim B can't win in P.
- Construct new profile Q:
  - Every voter in X puts C first, then A and B in same order as P.
  - Every voter in Y puts A and B first and second, in same order as P, then C third.
  - Voter v ranks A, then B, then C.

Dictatorial Control

Claim

The voter v has dictatorial control for A over B.

Proof.

Profile Q:

X		v	Y	
C	C	A	A	B
A	B	C	B	A
B	A	B	C	C
⋮	⋮	⋮	⋮	⋮

Dictatorial Control

Proof.

Profile 1

X	v	Y
C	C	A
A	A	B
B	B	C
⋮	⋮	⋮

Profile 2

X	v	Y
C	A	A
A	B	B
B	C	C
⋮	⋮	⋮

Profile Q

X		v	Y	
C	C	A	A	B
A	B	C	B	A
B	A	B	C	C
⋮	⋮	⋮	⋮	⋮

- C wins in profile 1 so by independence, B loses in profile Q.
- A wins in profile 2 so by independence, C loses in profile Q.
- By Pareto, only A can win in profile Q.
- By independence again, B can't win in original profile P.

Dictatorial Control

Claim

Some voter has dictatorial control for A over B.

Proof.

- Construct profiles where v is pivotal voter between A and C.
- Assume profile P where A is ranked over B.
- Build new profile Q where A must win.
- Independence shows that B can't win original profile P.

□

Lemma

If a social choice function with at least three candidates satisfies both Pareto and independence, then for any pair of candidates A and B, there is a voter with dictatorial control for A over B.

Arrow's Theorem

Theorem (Arrow)

If a social choice function with at least three candidates satisfies both Pareto and independence, then it must be a dictatorship.

Proof.

- By decisiveness lemma, must be decisive
- By other lemma, for any pair of candidates A and B, some voter v has dictatorial control for A over B.
- Claim: v also has dictatorial control for B over A.
- Claim: Same voter has dictatorial control over *every* pair
- Thus method is dictatorship.

Arrow's Theorem

Claim

If a social choice function with at least three candidates satisfies both Pareto and independence, then if v has dictatorial control for A over B, they also have dictatorial control for B over A.

Proof.

v	w	Others
A	B	A
B	A	B
⋮	⋮	⋮

- Suppose w has control for B over A.
- B can't win because of v
- A can't win because of w
- No one else can win by Pareto
- No winner. Contradiction!
- So v must have control for B over A. □

Arrow's Theorem

Claim

The same voter has dictatorial control over every pair .

Proof.

v	w	Others
C	B	C
A	C	B
B	A	A
⋮	⋮	⋮

- Suppose v has control between A and B, and w between B and C.
- B can't win because of v
- C can't win because of w
- No one else can win, by Pareto
- No winner. Contradiction!
- Same voter must have control over every pair.

□

Arrow's Theorem

Theorem (Arrow)

If a social choice function with at least three candidates satisfies both Pareto and independence, then it must be a dictatorship.

Proof.

- By decisiveness lemma, must be decisive
- By other lemma, for any pair of candidates A and B, some voter v has dictatorial control for A over B.
- v also has dictatorial control for B over A.
- v has dictatorial control over *every* pair
- Whichever candidate v ranks first wins
- The method is a dictatorship.

□

Tradeoffs

What have we learned?

- Can't be Condorcet and independent
- Can't be anonymous, neutral, and decisive
- Can't be Pareto, independent, and non-dictatorial.
- Can't hit half our criteria without being complicated!

Discussion Question

- What method would you want to use in an ideal world?
- What method should we use in *our* world?

Test on Thursday

Test Rules

- Plan to take the whole class
- Will have 6-7 questions
- Bring a one-sided handwritten note sheet
- You can bring a calculator but it probably won't be useful

Test Topics

- Two-candidate methods and criteria
- Multi-candidate methods and criteria
- Short proofs and counter-examples
- Drawing conclusions from criteria