Math 1007: Mathematics and Politics
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Mathematical Reasoning

Defining our terms

A big part of math is coming up with precise definitions so we're consistently talking about the same thing. This is really different from a lot of fields, where definitions can often be a lot fuzzier. Is the United States "a capitalist society"? Is it "a democracy"? Sort of but also it doesn't fit those idealized descriptions but also nothing really does. Categories in general are fuzzy.

In order to "do math" we need to write precise definitions so we don't have fuzzy categories; a thing either does, or does not, meet the definition. And then we can talk about what's true of *every* rule that falls into that category, because there are no edge cases that kinda sorta fall into the category but might be missing some important pieces.

But we also take the definition seriously. If we come up with a definition and it includes things we didn't think about originally, either we include those things in the category even if they "obviously shouldn't" count, or we have to start over and come up with a different definition. We'll do some of both in this course!

This is unlike the way we talk in the sciences, or the social sciences, or humanities, or every-day life. It is similar to the way we do *legal reasoning*, though. A criminal offense will have a specific list of elements, and you commit the crime if and only if you check off all those elements. A regulation will apply to everything meeting certain listed criteria, and sometimes that leads to weird results. Infamously, in California, a bee is a fish (for the purpose of certain environmental laws), because the law code defines "fish" to include "invertebrates" and bees are, in fact, invertebrates.

> Each of these statutes provides that covered species include "native species or subspecies of a bird, mammal, fish, amphibian, reptile, or plant[.]" This portion of the code, however, does not elaborate on what qualifies as a bird, mammal, fish, and so forth. Based only on the qualified species listed above, bees and other land-dwelling invertebrates would not receive protection under the law. The court looked elsewhere in the Fish and Game Code for definitions to help clarify whether bees may qualify for protection under CESA. Importantly, the section 45 of the code defines "fish" as "a wild fish, mollusk, crustacean, invertebrate, amphibian, or part, spawn, or ovum of any of those animals." (Emphasis added). According to the court, the term "invertebrate" under the definition of fish includes both aquatic and terrestrial invertebrates, such as bees. [**?**]

Mathematical reasoning is similar to legal (and philosophical) reasoning, in that we are necessarily concerned with the implications of the precise rules we have set up—and if we don't like those implications, we have to change the rules themselves.

Some of the definitions we're about to lay out will seem obvious. Others will seem quite strange! But it's important to have them all laid out clearly so that we are following the same rules and discussing the same ideas and procedures for the rest of the course. When we want to give a formal definition, we will lay it out in a block like this:

Definition 0.1. A *function* is a rule that assigns exactly one output to every valid input. We call the set of valid inputs the *domain* of the function, and the set of possible outputs the *codomain* or sometimes the *range*. A function must be deterministic, in that given the same input it will always yield the same output.

We will try to use "normal" English in our definitions as much as possible. This will sometimes lead to awkward phrasing or high levels of wordiness as we try to give precise and unambiguous definitions without resorting to too much technical, mathematical jargon. For instance, instead of definition 0.1 we could have written:

Definition 0.2 (Unnecessarily technical definition)**.** Let A,B be sets. We define a function $f : A \to B$ to be a set of ordered pairs $\{(a,b) : a \in A, b \in B\}$ such that for each a in A there is exactly one pair whose first element is a. We call A the domain and B the codomain of f.

That technically conveys the same information, but is much harder to read if you're not used to it; we'll avoid that sort of thing as much as we can.

Proving Theorems

Another big chunk of mathematical reasoning involves making very precise statements, and proving that they are true. We will often set these off in a block of text with a label. We use:

- "Theorem" for a major, important result
- "Proposition" for a less important result that we still care about
- "Lemma" for annoying technical results we mainly want in order to prove something else we actually care about
- "Corollary" for something that follows immediately from something we've already proven.

There are two interesting things going on here, besides the terminology. One is the idea of "proof". When we call something a theorem (or proposition or lemma), we mean that the statement is always true under all possible circumstances. That means we can *never* prove a theorem simply by providing examples of it being true. However, we can very easily *disprove* a theorem by showing a single example where it is false.

For instance, consider the statement "All swans are white". This has the form of a theorem; it claims that something is always true. How could we go about proving it? I could show you a lot of white swans, but no matter how many I show you, that doesn't prove that *all* swans are white; maybe most swans are white, but some are not. On the other hand, if you show me one black swan, you have proven that statement is false.

But it's very important that we really do mean always. Consider the statement "No swans are red". That's very close to being true. If you go out and survey a bunch of swans, you're unlikely to find a red one. But probably some swan, at some point, has accidentally gotten dyed red, and certainly it's possible that a swan *could* be dyed red. So the statement is false.

So how can we possibly have actual theorems? Here's a statement that works: "No black swan is white". That has to be true, always, because *if the swan is white, then it isn't black*. So a theorem almost always has a set of conditions or "hypotheses"; we conclude that any situation satisfying those hypotheses must also satisfy those conclusions.

For instance, the deficit is once again becoming a significant political issue. There are various ways we could try to tackle the deficit. Math can't tell you whether the deficit will increase or decrease. Math can't tell you if the deficit should increase or decrease. But we can say things like "If you increase revenue and decrease spending, then the deficit will go down." The hypotheses necessarily imply the conclusion. And conversely, if the deficit is going up, one of those hypotheses must be failing—either we are decreasing revenue or increasing spending.

Later in the course we will prove proposition 1.49, which says: "Any social choice function that is anonymous and neutral must not be decisive". Don't worry about what those words mean; we'll define them eventually. But the point is that while some methods are "decisive", whatever that means, not all are; and we can show that any time a method is "anonymous" and "neutral", it cannot be "decisive".

So maybe the most useful way to think about theorems is that they show you how something can fail. If we want to have a white swan, we need to look at swans that aren't black. If we want to have a decisive voting method, we need one that either isn't anonymous, or isn't neutral.

Math often gets presented as a set of timeless truths. And that's not *entirely* false. But it works because the truths are all conditional. We can't say that something always has to happen; but we can show that there are two things you can't ever get at the same time. We can show that some tradeoffs will always have to exist.