Calc 2

§1.5 Inverse Trig function

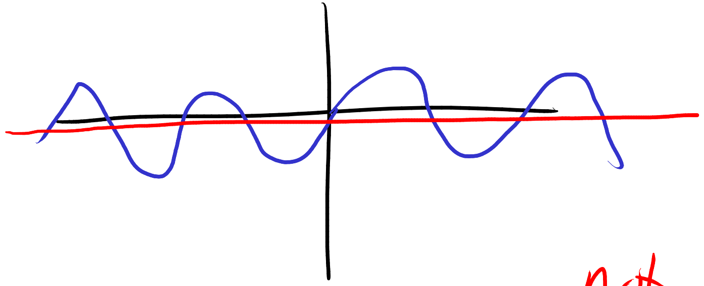
trig fns are transcendental

not $+, -, \times, \div$

Are they invertible?

$f(x) = \sin(x)$

Horizontal line test



hits more than once

not invertible

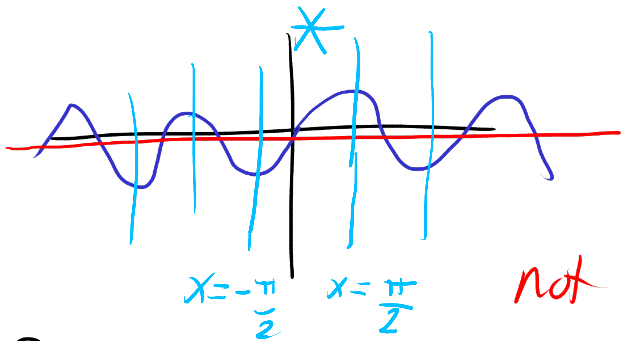
fn: 1 output for each input

1-1: 1 input for each output

$\sin(0) = 0, \ \sin(\pi) = 0$

$f(x) = \sin(x)$

Horizontal line test



hits more than once

not invertible

fn: 1 output for each input

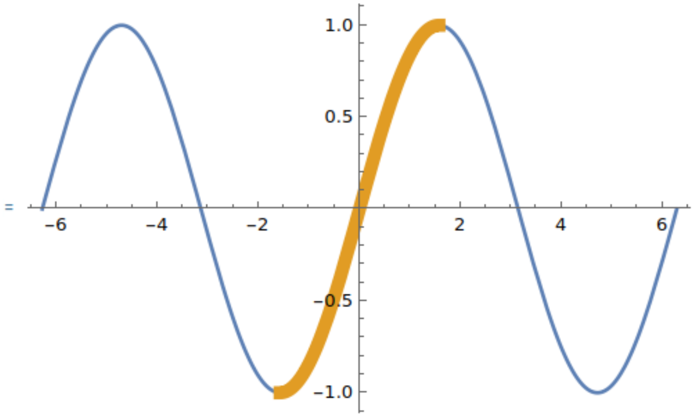
1-1: 1 input for each output

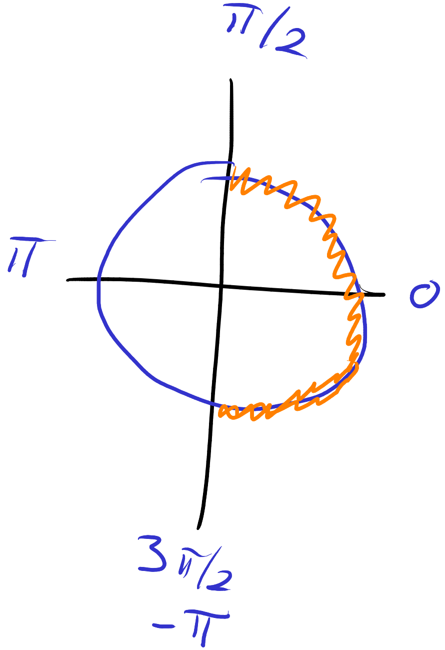
$\sin(0) = 0, \ \sin(\pi) = 0$

Q: $\sin(x) = 1$. What was x?

$\pi/2, \ 5\pi/2, \ -3\pi/2, \ 9\pi/2, \ldots$

Q: $\sin(x) = 1$ and $\underline{-\frac{\pi}{2} \le x \le \pi/2}$





$\sin(x)$ not 1-1

$\sin(x)$ on $[-\pi/2, \pi/2)$

is 1-1, and thus invertible.

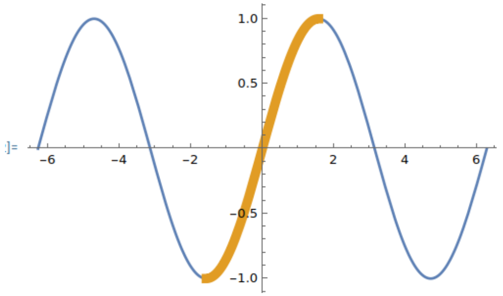
Dfn: for x in $[-1, 1]$, define

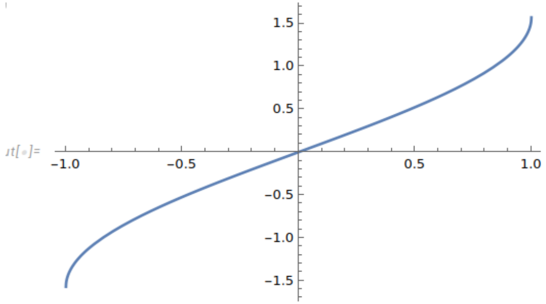
$\arcsin(x) = \sin^{-1}(x)$ to be

the inverse of $\sin(x)$ on $[-\pi/2, \pi/2]$

$\arcsin(x) = y$ where

- $\sin(y) = x$
- $-\pi/2 \le y \le \pi/2$



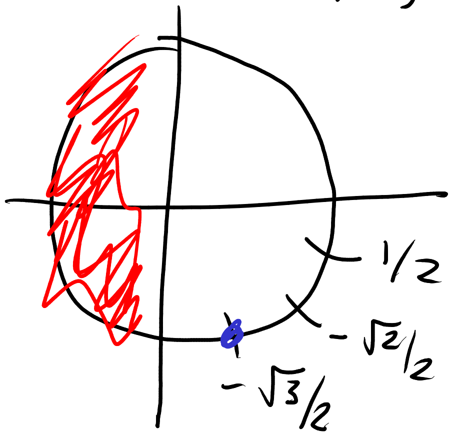


$\arcsin(x)$ has domain $[-1, 1]$

image/range $[-\pi/2, \pi/2]$

$\arcsin(1) = \pi/2$

$\arcsin(-\sqrt{3}/2) = -\pi/3$



~~$-2\pi/3$~~

~~$4\pi/3$~~

~~$5\pi/3$~~

$\arcsin(1/3)$

Q: solve $\sin(\theta) = 1/3$
not on unit circle
hard!

In[16]:= `ArcSin[1 / 3]`

Out[16]= $\text{ArcSin}\left[\frac{1}{3}\right]$

`N[ArcSin[1 / 3]]`

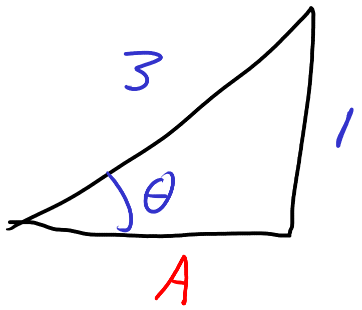
`0.339837`

Hard

$\cos(\arcsin(1/3))$

Solve $\sin(\theta) = 1/3$

$$\frac{opp}{hyp} = 1/3$$



Now find $\cos(\theta) = \frac{adj}{hyp}$

$$= \frac{\sqrt{8}}{3}.$$

$A^2 + 1^2 = 3^2$

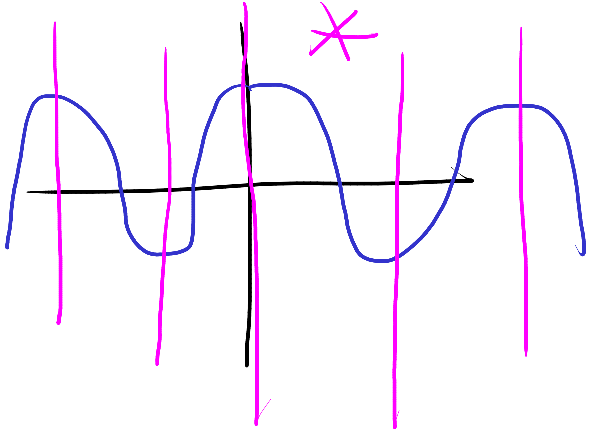
$A^2 + 1 = 9$

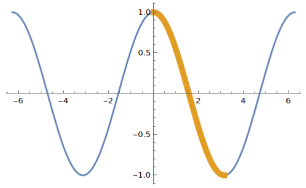
$A^2 = 8$

$A = \sqrt{8}$

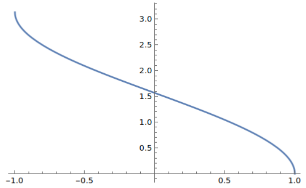
Important!

Inverse cosine





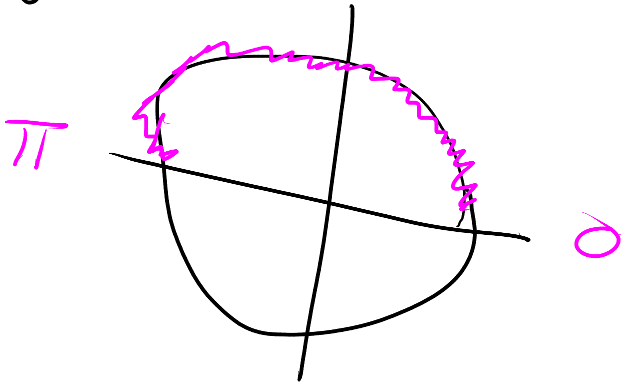
Plot[ArcCos[x], {x, -1, 1}]



for x in $[-1, 1]$, define

$\arccos(x) = \cos^{-1}(x) = y$ where

- $\cos(y) = x$
- $0 \le y \le \pi$



$\arccos(\sqrt{2}/2) = \pi/4$

Solve $\cos(y) = \sqrt{2}/2$

$\pi/4, -\pi/4$

$7\pi/4, -7\pi/4$

$\vdots$

2 notations

arcsin $\quad\quad$ $\sin^{-1}$

arccos $\quad\quad$ $\cos^{-1}$

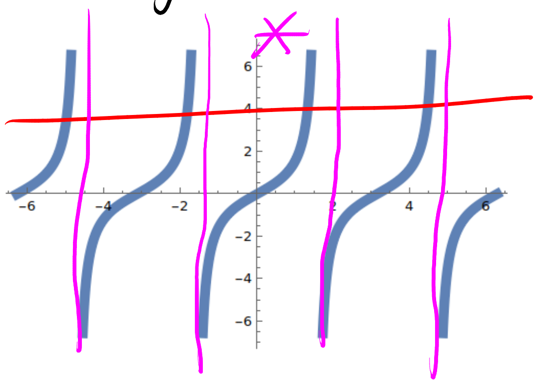
Much better $\quad\quad$ 'normal'

f^{-1}

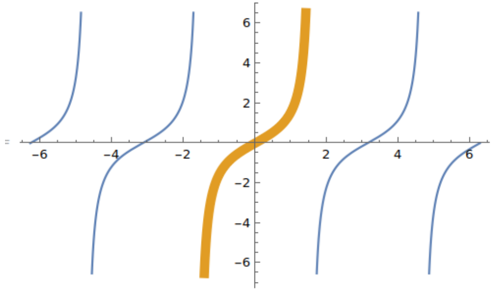
$$\sin^2(x) = (\sin(x))^2$$

$$\cancel{\sin^{-1}(x)} \stackrel{?}{=} \begin{cases} \cancel{(\sin(x))^{-1}} = \csc(x) \\ \arcsin(x) \end{cases}$$

Tangent



not 1-1



output all reals

tan(x) on $(-\pi/2, \pi/2)$

Dfn: if x is a real #, define

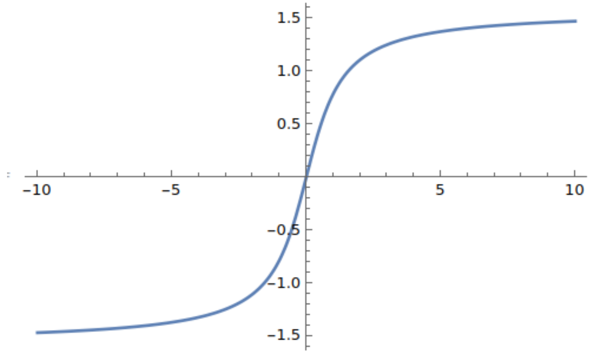
MOST IMPORTANT

$\arctan(x) = \tan^{-1}(x) = y$ where

- $\tan(y) = x$
- $-\pi/2 < y < \pi/2$

domain: all reals

range/image: $(-\pi/2, \pi/2)$



$$\lim_{x\to+\infty} \arctan(x) = \pi/2$$

$$\lim_{x\to-\infty} \arctan(x) = -\pi/2$$

$\frac{2}{\pi}\arctan(x)$: smooth transition from -1 to 1.

Derivatives of inverse trig

IFT: $(f^{-1})'(a) = \dfrac{1}{f'(f^{-1}(a))}$

Q: $\dfrac{d}{dx} \arcsin(x)$?

$$= \frac{1}{\sin'(\arcsin(x))}$$

$$= \frac{1}{\cos(\arcsin(x))}$$

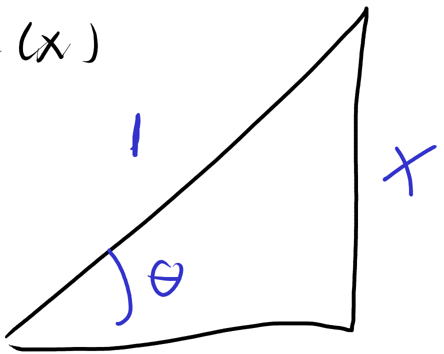
$$= \frac{1}{\cos(\theta)}$$

$$= \frac{1}{\sqrt{1-x^2}}.$$

Set $\theta = \arcsin(x)$

$\sin(\theta) = x$

$\dfrac{\text{opp}}{\text{hyp}} = \dfrac{x}{1}$



A

$\sqrt{1-x^2}$

$A^2 + x^2 = 1$

$A^2 = 1 - x^2$

$\cos(\theta) = \dfrac{\text{adj}}{\text{hyp}}$

$= \dfrac{\sqrt{1-x^2}}{1}$

$$\checkmark\ \frac{d}{dx}\arcsin(x) = \frac{1}{\sqrt{1-x^2}}$$

$$\frac{d}{dx}\arccos(x) = \frac{-1}{\sqrt{1-x^2}}$$

$$\checkmark\ \frac{d}{dx}\arctan(x) = \frac{1}{1+x^2} \checkmark$$

$$\frac{d}{dx}\ln|x| = \frac{1}{x}$$

$$\frac{d}{dx}\arctan(x) = \frac{1}{\tan'(\arctan(x))} = \frac{1}{\sec^2(\arctan(x))}$$

$$= \cos^2(\arctan(x))$$

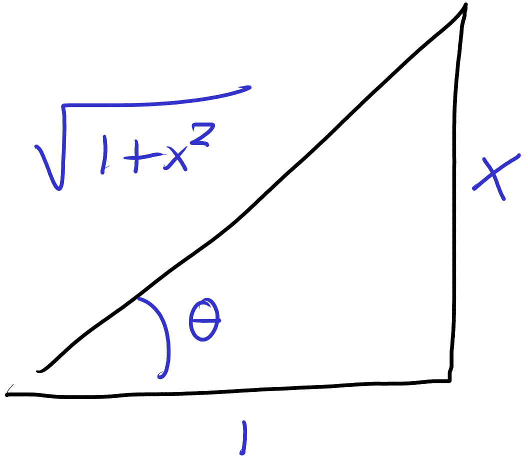
$$= \left(\frac{1}{\sqrt{1+x^2}}\right)^2 = \frac{1}{1+x^2}.$$

Q: $\cos(\arctan(x))$

$\theta = \arctan(x)$

$\tan(\theta) = x$

$$\frac{\text{opp}}{\text{adj}} = \frac{x}{1}$$



$$\cos(\arctan(x)) = \cos(\theta)$$

$$= \frac{\text{adj}}{\text{hyp}} = \frac{1}{\sqrt{1+x^2}}.$$

$e^x \longleftrightarrow$ trig

$\ln(x) \longleftrightarrow$ inverse trig

$$\frac{d}{dx}\arctan(e^x) = \frac{1}{1+(e^x)^2}\cdot e^x = \frac{e^x}{1+e^{2x}}$$

$$\frac{d}{dx}\arcsin(x^2+x) = \frac{1}{\sqrt{1-(x^2+x)^2}}\cdot(2x+1)$$

$$= \frac{2x+1}{\sqrt{1-(x^2+x)^2}}$$

Integrals using inverse trig

$$\int \frac{dx}{\sqrt{1-x^2}} = \arcsin(x) + C$$

$$\int \frac{dx}{1+x^2} = \arctan(x) + C$$

$$\int \frac{dx}{x^2+4} = \int \frac{1}{4} \cdot \frac{1}{x^2/4+1}\, dx = \int \frac{1}{4} \frac{1}{u^2+1} \cdot 2du$$

$$= \frac{1}{2}\int \frac{du}{u^2+1}$$

$$= \frac{1}{2} \arctan(u) + C$$

$$= \frac{1}{2} \arctan(x/2) + C$$

want u^2+1 on bottom

$$u^2+1 = \frac{x^2}{4}+1$$

$$u^2 = x^2/4$$

$$u = x/2$$

$$u = x/2$$

$$\frac{du}{dx} = 1/2$$

$$dx = 2du$$

$$\frac{d}{dx} \frac{1}{2}\arctan(x/2) = \frac{1}{2} \frac{1}{1+(x/2)^2} \cdot \frac{1}{2}$$

$$\int \frac{x}{\sqrt{1-x^4}}\,dx = \int \frac{\cancel{x}}{\sqrt{1-u^2}} \cdot \frac{du}{2\cancel{x}}$$

$u = 1 - x^4$ ✗

$\sqrt{1-x^4} = \sqrt{1-u^2}$

$u = x^2$

$\frac{du}{dx} = 2x$

$dx = \frac{du}{2x}$

$$= \frac{1}{2}\int \frac{du}{\sqrt{1-u^2}} = \frac{1}{2}\arcsin(u)$$

$$= \frac{1}{2}\arcsin(x^2).$$

One other idea

completing the square

$$x^2+2x+5 = (x+1)^2+4$$