§1.6 L'Hospital's Rule

Limits

Most limits are easy

$$\lim_{x\to 1} \frac{e^x-\ln(x^2)}{x^3+\sqrt[3]{x+7}} = \frac{e-\ln(1)}{1+\sqrt[3]{8}}$$

Just plug in

$$= \frac{e}{3}$$

(continuous)

$$\lim_{x\to 1} \frac{x^2-1 \xrightarrow{} 0}{x-1 \xrightarrow{} 0} \cancel{\neq \frac{0}{0}} = \lim_{x\to 1} \frac{\cancel{(x-1)}(x+1)}{\cancel{x-1}} = \lim_{x\to 1} x+1 = 2.$$

Indeterminate forms

$$\frac{0}{0}, \quad \frac{\infty}{\infty}$$

None of these are valid answers to anything

Indeterminate forms

$\frac{0}{0}$, $\frac{\infty}{\infty}$, $\infty \pm \infty$, $0 \cdot \infty$

1^{∞}, 0^{0}, ∞^{0}

None of these are valid answers to anything

$$e = \lim_{h \to 0} (1+h)^{1/h}$$

must do more work before you plug in.

Not #s, but not indeterminate

$\cancel{\frac{\infty}{0}}$ " $=$ " ∞

$\cancel{\frac{1}{0}}$ " $=$ " ∞

0^{∞} " $=$ " 0

2^{∞} " $=$ " 0

don't give as answers but they tell you what lim is no indet form: easy

indet form: maybe hard

few calc tools

work on transcendental fns

- factoring
- conjugate

What does transc fns?

- Small < approx

$$\lim_{x\to 0} \frac{\sin(x)}{x} = 1.$$

- Squeeze Thm

Thm (L'Hospital's Rule)

Suppose f, g are diff'able
$g'(x) \neq 0$ near a
(maybe at a)

not important

Suppose <u>either</u>

- $\lim_{x\to a} f(x) = \lim_{x\to a} g(x) = 0$
- $\lim_{x\to a} f(x) = \lim_{x\to a} g(x) = \pm\infty$

Very important

<u>Then</u>

$$\lim_{x\to a} \frac{f(x)}{g(x)} = \lim_{x\to a} \frac{f'(x)}{g'(x)}$$

if Right-Hand Side exists.

$$\lim_{x\to a} \frac{f(x) \nearrow 5}{g(x) \searrow 3} = \frac{5}{3}$$ easy

$$\lim_{x\to a} \frac{f(x) \nearrow 0}{g(x) \searrow 3} = 0$$ easy

$$\lim_{x\to a} \frac{f(x) \nearrow 3}{g(x) \searrow 0} = \pm\infty$$ medium

$$\lim_{x\to a} \frac{f(x) \nearrow 0}{g(x) \searrow 0}$$ maybe hard ~~✗✗~~

indet form L'H

Sketch of pf/

assume $f(a) = g(a) = 0$

$$f(x) \approx f(a) + f'(a)(x-a)$$

$$g(x) \approx g(a) + g'(a)(x-a)$$

$$\frac{f(x)}{g(x)} \approx \frac{\overset{0}{\cancel{f(a)}} + f'(a)(x-a)}{\underset{0}{\cancel{g(a)}} + g'(a)(x-a)}$$

$$= \frac{f'(a)\cancel{(x-a)}}{g'(a)\cancel{(x-a)}} = \frac{f'(a)}{g'(a)}$$

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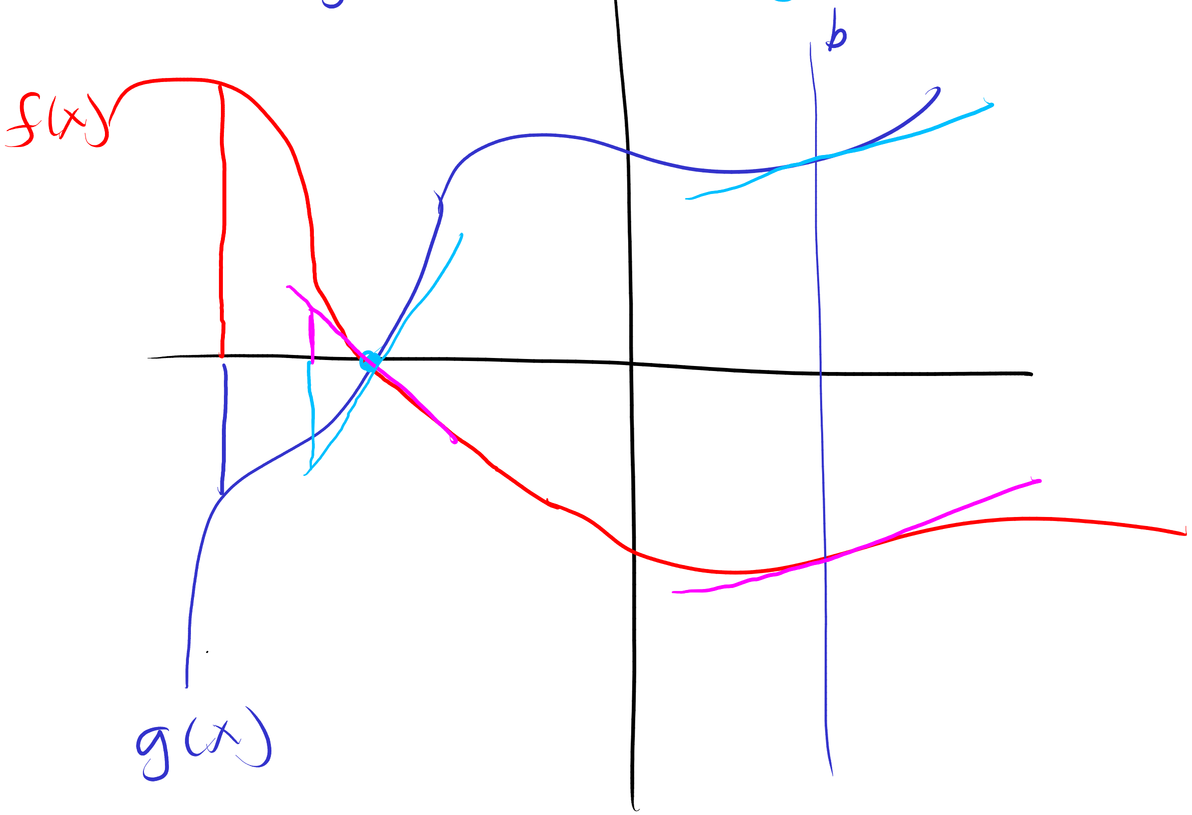
$$\frac{f(x)}{g(x)} \approx \frac{\text{slope } f}{\text{slope } g}$$



$$\lim_{x\to 3}\frac{x^2-4x+3 \nearrow 0}{x^2-2x-3 \searrow 0} \overset{L'H}{=} \lim_{x\to 3}\frac{2x-4 \nearrow 2}{2x-2 \searrow 4} = \frac{2}{4}$$

$$\lim_{x\to 0}\frac{1-\cos(x) \nearrow 0}{\sin(x) \searrow 0} \overset{L'H}{=} \lim_{x\to 0}\frac{\sin(x) \nearrow 0}{\cos(x) \searrow 1} = \frac{0}{1} = 0$$

Calc I: $\cdot \frac{1+\cos(x)}{1+\cos(x)}$

$$\lim_{x\to 1}\frac{\ln(x) \nearrow 0}{x-1 \searrow 0} \overset{L'H}{=} \lim_{x\to 1}\frac{1/x \nearrow 1}{1 \searrow 1} = \frac{1}{1} = 1.$$

No quotient Rule!

Show you checked that L'H applies

Don't write

$= \cancel{0/0}$

- $\lim_{x\to 0}\frac{e^x-1-x}{x^2}\begin{matrix}\nearrow 0\\ \searrow 0\end{matrix}\overset{L'H}{=}\lim_{x\to 0}\frac{e^x-1}{2x}\begin{matrix}\nearrow 0\\ \searrow 0\end{matrix}\overset{L'H}{=}\lim_{x\to 0}\frac{e^x}{2}\begin{matrix}\nearrow 1\\ \searrow 2\end{matrix}=\frac{1}{2}$

Can also use for $\frac{\infty}{\infty}$

- $\lim_{x\to+\infty}\frac{x^2+5x+3}{x^2+7x-2}\begin{matrix}\nearrow\infty\\ \searrow\infty\end{matrix}\overset{L'H}{=}\lim_{x\to+\infty}\frac{2x+5}{2x+7}\begin{matrix}\nearrow\infty\\ \searrow\infty\end{matrix}\overset{L'H}{=}\lim_{x\to+\infty}\frac{2}{2}=1.$

- $\lim_{x\to+\infty}\frac{\ln(x)}{x}\begin{matrix}\nearrow+\infty\\ \searrow+\infty\end{matrix}\overset{L'H}{=}\lim_{x\to+\infty}\frac{1/x}{1}=\lim_{x\to+\infty}\frac{1}{x}\begin{matrix}\nearrow 1\\ \searrow\infty\end{matrix}=0 \leftarrow$ Important for CS

- $\lim_{x\to+\infty}\frac{e^x}{x}\begin{matrix}\nearrow\infty\\ \searrow\infty\end{matrix}\overset{L'H}{=}\lim_{x\to+\infty}\frac{e^x}{1}\begin{matrix}\nearrow+\infty\\ \searrow 1\end{matrix}=+\infty$

Warning examples

$$\lim_{x\to\pi}\frac{\sin(x)\nearrow 0}{1-\cos(x)\searrow 2}\ \cancel{\overset{L'H}{=}}\ \lim_{x\to\pi}\frac{\cos(x)\nearrow -1}{\sin(x)\searrow 0}=\pm\infty$$

$$\frac{-1}{\text{small}\ \#}=\text{big}\ \#$$

(maybe negative)

$$=\frac{0}{2}=0$$

$$\lim_{x\to 0}\frac{e^x-1-x\nearrow 0}{x^3\searrow 0}\overset{L'H}{=}\lim_{x\to 0}\frac{e^x-1\nearrow 0}{3x^2\searrow 0}\overset{L'H}{=}\lim_{x\to 0}\frac{e^x\nearrow 1}{6x\searrow 0}\ \cancel{\overset{L'H}{=}}\ \lim_{x\to 0}\frac{e^x}{6}=\frac{1}{6}.$$

WRONG

$$=\pm\infty.$$

Bonus:

$$\lim_{x\to+\infty}\frac{x\to\infty}{\sqrt{x^2+1}\searrow\infty}$$

Can solve in calc I.

L'H does not help.

$$\lim_{x\to 0}\frac{e^x\nearrow 1}{e^x\searrow 1}=1.$$

how to handle other IFS?

Rewrite as $\frac{f}{g}$

$$\lim_{x\to 1} x^{1/1-x} = e^{-1} = 1/e.$$

1^{∞}

$$y = x^{\frac{1}{1-x}}$$

$$\ln|y| = \ln\left|x^{\frac{1}{1-x}}\right| = \frac{1}{1-x}\ln|x|$$

$$\lim_{x\to 1} \frac{\ln|x| \to 0}{1-x \to 0} \overset{L'H}{=} \lim_{x\to 1} \frac{1/x}{-1} = -1.$$

$$\lim_{x\to 1} \ln|y| = -1 \Rightarrow \lim_{x\to 1} y = \lim_{x\to 1} e^{\ln|y|} = e^{-1}$$