Math 1232: Single-Variable Calculus 2
George Washington University   Spring 2026
Recitation 10

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March 25, 2026

Problem 1 (Infinite Decimals)**.** We want to find a rational representation of the infinite decimal $0.\overline{47}$. That is, we want to write $0.\overline{47} = \frac{p}{q}$ for integers p, q.

(a) First, what happens if we multiply $0.\overline{47}$ by 100?

(b) Using part (a), what can you tell about $(99) \cdot 0.\overline{47}$?

(c) Give a rational representation of $0.\overline{47}$.

(d) Now let's take a different approach. Write $0.\overline{47}$ as an infinite series.

(e) What kind of series is this? Can you use that fact to find a rational representation of $0.\overline{47}$?

(f) Now use the same logic to find a rational representation of $2.\overline{63}$.

Solution:

(a) $0.\overline{47} \cdot 100 = 47.\overline{47}$.

(b)

$$\begin{aligned} 99 \cdot 0.\overline{47} &= 100 \cdot 0.\overline{47} - 0.\overline{47} \\ &= 47.\overline{47} - 0.\overline{47} = 47 \end{aligned}$$

(c) Thus $0.\overline{47} = \frac{47}{99}$.

(d) We can write $0.\overline{47} = \sum_{k=1}^{\infty} 47 \cdot 100^{-k}$.

(e) This is a geometric series with $a = \frac{47}{100}$ and $r = \frac{1}{100}$. Thus

$$0.\overline{47} = \sum_{k=1}^{\infty} 47 \cdot 100^{-k} = \frac{47/100}{1 - 1/100} = \frac{47/100}{99/100} = \frac{47}{99}.$$

(f) We can ignore the 2 until later. We can see

$$\begin{aligned} 0.\overline{63} &= \sum_{k=1}^{\infty} 63 \cdot 100^{-k} \\ &= \frac{63/100}{1 - 1/100} = \frac{63/100}{99/100} \\ &= \frac{63}{99} \\ 2.\overline{63} &= 2 + \frac{63}{99} = \frac{261}{99}. \end{aligned}$$

Problem 2. Consider the series $\displaystyle\sum_{n=1}^{\infty} \frac{n^2 - 1}{n^4}$.

(a) Can you find a simpler series that's larger than this? One that's smaller?

(b) Do the series you found in part (a) converge or diverge? Why? What does that tell you about our original series?

(c) Now consider the series $\displaystyle\sum_{n=1}^{\infty} \frac{n^2 + 1}{n^4}$. Can you find a simpler series that's larger than this? One that's smaller?

(d) Do the series you found in part (c) converge or diverge? What does that tell you about the convergence of the original series?

(e) Now let's not worry about bigger or smaller. What is the simpler series that part (c) *kinda looks like*?

(f) How can we use that to figure out if the series in part (c) converges or diverges?

Solution:

(a) It's pretty easy to see that $\frac{n^2-1}{n^4} < \frac{n^2}{n^4} = \frac{1}{n^2}$.

You can find a smaller series but not really reasonably. Like obviously you can say that $\frac{n^2-1}{n^4} > \frac{n^2-2}{n^4}$ but that doesn't really accomplish anything. Stretching a bit more, you could see that $\frac{n^2-1}{n^4} > \frac{n^2}{n^5} = \frac{1}{n^3}$ but that's a very different series!

(b) We know that $\sum \frac{1}{n^2}$ converges, by the p-series test with $p = 2$. That means our original series converges by the comparison test.

We also know that $\sum \frac{1}{n^3}$ converges, by the p-series test with $p = 3$. But that wouldn't actually help us with anything; showing that our series is *bigger* than a *convergent* series doesn't help.

(c) In this case we see that $\frac{n^2+1}{n^4} > \frac{n^2}{n^4} = \frac{1}{n^2}$. We can also probably convince ourselves that $\frac{n^2+1}{n^4} < \frac{n^3}{n^4} = \frac{1}{n}$, at least for $n > 2$ or so.

(d) We still know that $\frac{1}{n^2}$ converges by the p-series test. But now that doesn't help us; our series is bigger than a convergent series, which isn't useful.

Conversely, we know $\sum \frac{1}{n}$ diverges, by the p-series test with $p = 1$ or just by "it's the harmonic series, which diverges". This also doesn't help us! Our series is smaller than a divergent series, but that could be anything. This isn't helpful.

(Now it's possible to be a bit more clever here, and argue something like $\frac{n^2+1}{n^4} < \frac{n^{2.5}}{n^4} = \frac{1}{n^{1.5}}$ which converges. But that tends to be awkward and sometimes it's hard to figure out how to make an argument like that work.)

(e) But really, easier than doing any of that is just to notice that $\frac{n^2+1}{n^4}$ is almost the same thing as $\frac{n^2}{n^4} = \frac{1}{n^2}$.

(f) The comparison test doesn't work here because the inequality goes in the wrong direction. But this is the exact case for the limit comparison test, which doesn't need to worry about an inequality. We compute that

$$\lim_{n\to\infty} \frac{n^2 + 1/n^4}{1/n^2} = \lim_{n\to\infty} \frac{n^4 + n^2}{n^4} = \lim_{n\to\infty} 1 + \frac{1}{n^2} = 1.$$

Thus, since $\sum \frac{1}{n^2}$ converges by the p-series test, we know that $\sum_{n=1}^{\infty} \frac{n^2+1}{n^4}$ converges by the Limit Comparison Test.

Problem 3. For each of the following series, write a careful argument showing either that it converges or that it diverges. Think about exactly what test you want to use and why.

(a) $\displaystyle\sum_{n=2}^{\infty} \frac{5n^3 - 2}{3n^5 - n}$

(b) $\displaystyle\sum_{n=2}^{\infty} \frac{n^3 \ln(n) + 1}{n^4 - 7}$.

(c) $\sum_{n=2}^{\infty} \frac{1}{n\ln(n)}$

Solution:

(a) This is a pile of polynomials, so it'll be simplest to use the limit comparison test. It looks like $\frac{n^3}{n^5} = \frac{1}{n^2}$, so we compute

$$\lim_{n\to\infty} \frac{\frac{5n^3-2}{3n^5-n}}{\frac{1}{n^2}} = \lim_{n\to\infty} \frac{5n^5-2n^3}{3n^5-n} = 5/3.$$

This is a real finite non-zero limit. Then since $\sum_{n=2}^{\infty} \frac{1}{n^2}$ converges as a p-series, our original series converges by the Limit Comparison Test.

(b) This doesn't look at all geometric, but also isn't just polynomials, so we hope the regular comparison test works. This looks kinda like $\frac{n^3}{n^4} = \frac{1}{n}$. And in fact we see $n^3\ln(n)+1 > n^3\ln(n) > n^3$ as long as $n > 2$, and $n^4 - 7 < n^4$. So we have

$$\frac{n^3\ln(n)+1}{n^4-7} > \frac{n^3}{n^4} = \frac{1}{n}.$$

Since $\sum_{n=2}^{\infty} \frac{1}{n}$ diverges by the p-series test, we know that our original series diverges by the comparison test.

(c) This one has no obvious ideas for approaching it. Or rather, it's "obvious" to try to set up a comparison test, but it doesn't work; we can see that

$$\frac{1}{n\ln(n)} < \frac{1}{n}$$

but since $\sum \frac{1}{n}$ diverges that doesn't tell us anything.

In the absence of better ideas, we have to use the integral test. We want to compute $\int_2^{+\infty} \frac{dx}{x\ln(x)}$. If we set $u = \ln(x)$ we get $du = \frac{1}{x}\,dx$ and then we have

$$\begin{aligned}
\int_2^{+\infty} \frac{dx}{x\ln(x)} &= \lim_{t\to+\infty} \int_2^t \frac{dx}{x\ln(x)} \\
&= \lim_{t\to+\infty} \int_{\ln(2)}^{\ln(t)} \frac{du}{u} \\
&= \lim_{t\to+\infty} \ln(u)\Big|_{\ln(2)}^{\ln(t)} = \lim_{t\to+\infty} \ln(\ln(t)) - \ln(\ln(2)) = +\infty.
\end{aligned}$$

Thus the integral diverges, and by the integral test the series $\sum_{n=2}^{\infty} \frac{1}{n\ln(n)}$ also diverges.

Problem 4 (Bonus)**.** Does the series $\sum_{n=1}^{\infty} \frac{\sin^2(n^2+e^n)}{n^2}$ converge or diverge?

Solution: We know that $0 \leq \sin^2(n^2 + e^n) \leq 1$, so

We have that $\frac{\sin^2(n^2+e^n)}{n^2-n} \leq \frac{1}{n^2}$, and $\sum_{n=1}^{\infty} \frac{1}{n^2}$ converges by the $p-$ series test. So by the comparison test the series $\sum_{n=1}^{\infty} \frac{\sin^2(n^2+e^n)}{n^2-n}$ converges.