Math 1232 Spring 2026
Single-Variable Calculus 2
Mastery Quiz 11
Due Thursday, April 9

This week's mastery quiz has three topics. Everyone should submit work on both M4. If you have a 2/2 on S8, or a 4/4 on M3, you don't need to submit them again. This is the last quiz with S8.

Don't worry if you make a minor error, but try to demonstrate your mastery of the underlying material. Remember that you are trying to demonstrate that you understand the concepts involved. For all these problems, justify your answers and explain how you reached them. Do not just write "yes" or "no" or give a single number.

Feel free to consult your notes, but please **don't discuss the actual quiz questions with other students in the course**.

Please turn this quiz in class on Thursday. You may print this document out and write on it, or you may submit your work on separate paper; in either case make sure your name and recitation section are clearly on it.

Topics on This Quiz

- Major Topic 3: Series Convergence
- Major Topic 4: Taylor Series
- Secondary Topic 8: Power Series

Name:

Recitation Section:

M3: Series Convergence

(a) Analyze the convergence of the series $\displaystyle\sum_{n=1}^{\infty} \frac{4n^3+1}{n^4+n+3}$

Solution: We have

$$\int_1^{\infty} \frac{4x^3+1}{x^4+x+3}\,dx = \lim_{t\to\infty}\int_1^{t} \frac{4x^3+1}{x^4+x+3}\,dx = \lim_{t\to\infty} \ln(x^4+x+3)\big|_1^t$$
$$= \lim_{t\to\infty} \ln(t^4+t+3) - \ln(5) = \infty.$$

This diverges, so by the integral test the series $\sum_{i=1}^{\infty} \frac{4n^3+1}{n^4+n+3}$ diverges.

We can also try to use the Comparison Test here, but it's a little tricky, because $\frac{4n^3+1}{n^4+n+3} < \frac{4}{n}$, and that doesn't help because $\sum_{n=1}^{\infty} \frac{4}{n} = \infty$. If we want to do comparison, we can try to argue that while $\frac{4n^3+1}{n^4+n+3} < \frac{4}{n}$, it's also true that $\frac{4n^3+1}{n^4+n+3} > \frac{1}{n}$. But that's not super obvious and would take some work.

Or we could use the Limit Comparison Test, and argue that

$$\lim_{n\to\infty} \frac{\frac{4n^3+1}{n^4+n+3}}{\frac{1}{n}} = \lim_{n\to\infty} \frac{4n^4+n}{n^4+n+3} = 4$$

is a finite non-zero limit. Since $\sum_{n=1}^{\infty} \frac{1}{n}$ diverges (by the p-series test, or because it's the harmonic series), we know by the Limit Comparison Test that $\sum_{n=1}^{\infty} \frac{4n^3+1}{n^4+n+3}$ diverges.

(b) $\displaystyle\sum_{n=4}^{\infty} \frac{(-1)^n}{(n^2/5)+3n}$

Solution: This clearly converges by the alternating series test, since $\lim_{n\to\infty} \frac{1}{n^2/5-3n} = 0$, but does it absolutely converge? The ratio test won't work; if we work it out we'll get a limit of 1. But we have

$$\sum_{n=4}^{\infty} \left| \frac{(-1)^n}{n^2/5+3n} \right| = \sum_{n=4}^{\infty} \frac{1}{n^2/5+3n},$$

so we can use the Limit Comparison Test to $\frac{1}{n^2}$. We compute

$$\lim_{n\to\infty} \frac{\frac{1}{n^2/5+3n}}{\frac{1}{n^2}} = \lim_{n\to\infty} \frac{n^2}{n^2/5+3n} = 1/5.$$

This is a nonzero real number, so since $\sum_{n=4}^{\infty} \frac{1}{n^2}$ converges, by the Limit Comparison Test, $\sum_{n=4}^{\infty} \frac{1}{n^2/5+3n}$ converges. Thus our original series converges absolutely. (And thus we don't actually need to check for whether the alternating series test applies.)

(c) $\sum_{n=1}^{\infty}(-1)^n \frac{n^2-1}{n^3+2}$

Solution: First we check absolute convergence, by looking at

$$\sum_{n=1}^{\infty}\left|(-1)^n \frac{n^2-1}{n^3+2}\right| = \sum_{n=1}^{\infty}\frac{n^2-1}{n^3+2}.$$

Here it would be hard to use the regular comparison test. It's true that $\frac{n^2-1}{3n^3+2} \leq \frac{1}{n}$, but since this says it's less than a divergent series, it doesn't really help.

Instead, we limit compare to $\frac{1}{n}$. We have

$$\lim_{n\to\infty} \frac{n^2-1/3n^3+1}{1/n} = \lim_{n\to\infty} \frac{n^3-3n}{3n^3+1} = \frac{1}{3}.$$

Since this is a finite non-zero number, the two series have the same convergence behavior. (It doesn't matter that $1/3 < 1$; all that matters is it's finite and non-zero.) Thus, since $\frac{1}{n}$ diverges, we know that $\sum_{n=1}^{\infty}\left|(-1)^n \frac{n^2-1}{n^3+2}\right|$ also diverges.

So now we should check for conditional convergence. This is an alternating series, and $\lim_{n\to\infty} \frac{n^2-1}{n^3+2} - 0$. so by the alternating series test, our series converges; thus it converges conditionally.

M4: Taylor Series

(a) Write a power series expression for $\frac{x}{2+x^2}$ centered a 0. What is the radius of convergence?

Solution: We know that

$$\frac{1}{2-x} = \frac{1}{2}\frac{1}{1-x/2} = \frac{1}{2}\sum_{n=0}^{\infty}(x/2)^n$$

$$\frac{1}{2+x^2} = \frac{1}{2}\frac{1}{1-(-x^2/2)} = \frac{1}{2}\sum_{n=0}^{\infty}(-x^2/2)^n = \sum_{n=0}^{\infty}\frac{(-1)^n}{2^{n+1}}x^{2n}$$

$$\frac{x}{2+x^2} = \sum_{n=0}^{\infty}\frac{(-1)^n}{2^{n+1}}x^{2n+1}.$$

The radius of convergence is $\sqrt{2}$. We can figure that out by reasoning from the geometric series: the radius of convergence for the geometric series is 1, so it converges for $-1 < x^2/2 < 1$ or $-2 < x^2 < 2$ or $-\sqrt{2} < x < \sqrt{2}$. Or we can use the ratio test:

$$\lim_{n\to\infty} \left| \frac{x^{2n+3}/2^{n+2}}{x^{2n+1}/2^{n+1}} \right| = \lim_{n\to\infty} \frac{|x|^2}{2}$$

and thus it converges when $x^2/2 < 1$.

(b) Let $f(x) = e^{x^2}$. Use *the definition of a Taylor series* to find $T_4(x, 0)$ for this function. (That is, find the terms up through the degree-four term.)

Solution:

$$f(x) = e^{x^2} \qquad f(0) = 1$$

$$f'(x) = 2xe^{x^2} \qquad f'(0) = 0$$

$$f''(x) = 2e^{x^2} + 4x^2e^{x^2} \qquad f''(0) = 2f'''(x) \quad = 4xe^{x^2} + 8xe^{x^2} + 8x^3e^{x^2} f'''($$

$$f^{(4)} = 12e^{x^2} + 24x^2e^{x^2} + 24x^2e^{x^2} + 16x^4e^{x^2} \quad f^{(4)}(0) = 12.$$

So we have

$$T_4(x, \pi) = 1 + \frac{2}{2}x^2 + \frac{12}{4!}x^4 = 1 + x^2 + x^4/2.$$

(c) If $f(x) = \sum_{n=0}^{\infty} 2^n n^3 (x-2)^n$, compute $\frac{d}{dx}f(x)$ and $\int f(x)\,dx$.

Solution:

$$\frac{d}{dx}f(x) = \sum_{n=0}^{\infty} 2^n n^4 (x-2)^{n-1}$$

$$\text{(or better)} = \sum_{n=1}^{\infty} 2^n n^4 (x-2)^{n-1}$$

$$\int f(x)\,dx = \sum_{n=0}^{\infty} \frac{2^n n^3}{n+1}(x-2)^{n+1} + C$$

$$\text{(or)} = \sum_{n=1}^{\infty} \frac{2^{n-1}(n-1)^3}{n}(x-2)^n + C.$$

S8: Power Series

(a) Find the radius of convergence and the interval of convergence of $\sum_{n=0}^{\infty} \frac{2^n}{n^2+n} x^n$.

Solution: We use the ratio test.

$$\lim_{n\to\infty}\left|\frac{2^{n+1}x^{n+1}/(n+1)^2+n+1}{2^n x^n/n^2+n}\right| = \lim_{n\to\infty} 2|x|\frac{n^2+3n+2}{n^2+n}$$
$$= 2|x|.$$

So we need $2|x| < 1$ or $-1 < 2x < 1$, or $-1/2 < x < 1/2$ We need to have x in the interval $(0-1/2, 0+1/2)$, so the radius is $1/2$.

To find the interval we need to check the endpoints. We see

$$\sum_{n=0}^{\infty}\frac{2^n}{n^2+n}(1/2)^n \sum_{n=0}^{\infty}\frac{1}{n^2+n}$$

converges by comparison to a p-series

$$\sum_{n=0}^{\infty}\frac{2^n}{n^2+n}(-1/2)^n = \sum_{n=0}^{\infty}\frac{(-1)^n}{n^2+n}$$

converges by the alternating series test

Thus the interval of convergence is $[-1/2, 1/2]$.

(b) Find the radius of convergence and the interval of convergence of $\sum_{n=0}^{\infty}\frac{n}{5^n}(x-3)^n$.

Solution: We use the ratio test.

$$\lim_{n\to\infty}\left|\frac{(n+1)(x-3)^{n+1}/5^{n+1}}{(n)(x-3)^n/5^n}\right| = \lim_{n\to\infty}\left|\frac{n+1}{n}\frac{x-3}{5}\right|$$
$$= |x-3|/5\lim_{n\to\infty}\frac{n+1}{n} = |x-3|/5.$$

So we need $|x-3|/5 < 1$ or $-5 < x-3 < 5$, or $-2 < x < 8$ or $3-5 < x < 3+5$. So the radius is 5.

To find the interval we need to check the endpoints. We see

$$\sum_{n=0}^{\infty}\frac{n}{5^n}5^n = \sum_{n=0}^{\infty} n$$

diverges by divergence or p-series test

$$\sum_{n=0}^{\infty}\frac{n}{5^n}(-5)^n = \sum_{n=0}^{\infty}(-1)^n n$$

diverges by divergence test

Thus the interval is $(-2, 8)$.