

Math 1232 Spring 2026
Single-Variable Calculus 2
Mastery Quiz 12
Due Thursday, April 16

This week's mastery quiz has three topics. Everyone should submit work on both M4 and S9. If you have a 4/4 on M3, you don't need to submit it again. This is the last quiz with M3.

Don't worry if you make a minor error, but try to demonstrate your mastery of the underlying material. Remember that you are trying to demonstrate that you understand the concepts involved. For all these problems, justify your answers and explain how you reached them. Do not just write "yes" or "no" or give a single number.

Feel free to consult your notes, but please **don't discuss the actual quiz questions with other students in the course.**

Please turn this quiz in class on Thursday. You may print this document out and write on it, or you may submit your work on separate paper; in either case make sure your name and recitation section are clearly on it.

Topics on This Quiz

- Major Topic 3: Series Convergence
- Major Topic 4: Taylor Series
- Secondary Topic 9: Applications of Taylor Series

Name:

Recitation Section:

M3: Series Convergence

- (a) Analyze the convergence of the series $\sum_{n=1}^{\infty} (-1)^n \frac{n^2 + n - 3}{n^2 4^n}$

Solution: We use the ratio test. We have

$$\begin{aligned} L &= \lim_{n \rightarrow \infty} \left| \frac{(-1)^{n+1} (n+1)^2 + n + 1 - 3 / (n+1)^2 4^{n+1}}{(-1)^n n^2 + n - 3 / n^2 4^n} \right| \\ &= \lim_{n \rightarrow \infty} \left| \frac{(n^2 + 2n - 1)n^2 4^n}{(n+1)^2 4^{n+1} (n^2 + n - 3)} \right| \\ &= \lim_{n \rightarrow \infty} \left| \frac{n^4 + 2n^3 - n^2}{4n^4 + 12n^3 - 20n - 15} \right| = \frac{1}{4} < 1. \end{aligned}$$

So by the ratio test this converges absolutely..

- (b) Analyze the convergence of $\sum_{n=1}^{\infty} \frac{(-1)^n \sqrt{n}}{2n+3}$

Solution: This is an alternating series. Since the terms $\frac{\sqrt{n}}{2n+3}$ tend to zero as n goes to infinity, this converges by the alternating series test.

However, it doesn't absolutely converge. If we look at the absolute value series, we have $\sum_{n=1}^{\infty} \frac{\sqrt{n}}{2n+3}$. You can see this doesn't converge in a couple ways. The integral test isn't super plausible here. You can do a comparison test to $\frac{1}{\sqrt{n}}$: this is larger than $\frac{1}{3\sqrt{n}}$ for large n , and $\frac{1}{3\sqrt{n}}$ diverges. (note: this is *not* larger than $\frac{1}{\sqrt{n}}$!)

It may be easier to use the limit comparison test, though. We have

$$\lim_{n \rightarrow \infty} \frac{\sqrt{n}/(2n+3)}{1/\sqrt{n}} = \lim_{n \rightarrow \infty} \frac{n}{2n+3} = 1/2.$$

Since the series $\sum \frac{1}{\sqrt{n}}$ diverges, by the limit comparison test, $\sum \frac{\sqrt{n}}{2n+3}$ diverges, and thus our series does not converge absolutely.

- (c) Analyze the convergence of $\sum_{n=1}^{\infty} (-1)^n \frac{\sin(1/n)}{n^2}$.

Solution: We first check absolute convergence, so check the convergence of $\sum_{n=1}^{\infty} \frac{\sin(1/n)}{n^2}$.

You can't really use the limit comparison test here, because the obvious comparison $\sum 1/n^2$ leads to you needing to compute $\lim_{n \rightarrow \infty} \sin(1/n)$, which doesn't exist.

But you can use the usual comparison test. We know that

$$\begin{aligned}\sin(1/n) &\leq 1 \\ \frac{\sin(1/n)}{n^2} &\leq \frac{1}{n^2}\end{aligned}$$

Since $\sum 1/n^2$ converges, by the comparison test, $\sum_{n=1}^{\infty} \frac{\sin(1/n)}{n^2}$ converges. So our original series $\sum_{n=1}^{\infty} (-1)^n \frac{\sin(1/n)}{n^2}$ converges absolutely.

M4: Taylor Series

- (a) Using series we already know, write down a formula for the (infinite) Taylor series for $x(8+x)^{5/3}$, and then write down the degree-four polynomial explicitly.

Solution: We can take this from the binomial series for $(1+x)^\alpha$. So we have

$$\begin{aligned}(1+x/8)^{5/3} &= \sum_{n=0}^{\infty} \binom{5/3}{n} (x/8)^n \\ (8+x)^{5/3} &= 32(1+x/8)^{5/3} = \sum_{n=0}^{\infty} \binom{5/3}{n} 2^{5-3n} x^n \\ x(8+x)^{5/3} &= \sum_{n=0}^{\infty} \binom{5/3}{n} 2^{5-3n} x^{n+1} \\ T_4(x,0) &= 32x + 5/3 \cdot 4x^2 + \frac{(\binom{5/3}{2/3})}{2} \cdot \frac{1}{2} x^3 + \frac{(\binom{5/3}{2/3})(\binom{-1/3}{1})}{6} \frac{1}{16} x^4 \\ &= 32x + \frac{20}{3} x^2 + \frac{5}{18} x^3 - \frac{5}{1296} x^4.\end{aligned}$$

- (b) Estimate the error if you use $T_4(x) = 1 - \frac{x^2}{2} + \frac{x^4}{4!}$ to approximate $g(x) = \cos(x)$ at $x = -.5$.

Solution: We can use the remainder theorem. Since $g^{(5)}(x) = -\sin(x)$, we have

$$\begin{aligned}|R_4(x)| &= \left| \frac{f^{(5)}(z)}{5!} x^5 \right| = \left| \frac{\sin(z)}{5!} x^5 \right| \leq \frac{|x|^5}{5!} \\ &= \frac{.5^5}{120} = \frac{1}{3840} \approx .00026.\end{aligned}$$

- (c) Let $f(x) = \sin(x)$. Use the definition of a Taylor series to find $T_3(x, \pi/3)$ (centered at $\pi/3$) for this function. (That is, find the terms up through the degree-three term.)

Solution:

$$\begin{array}{ll}
 f(x) = \sin(x) & f(\pi/3) = \sqrt{3}/2 \\
 f'(x) = \cos(x) & f'(\pi/3) = 1/2 \\
 f''(x) = -\sin(x) & f''(\pi/3) = -\sqrt{3}/2 \\
 f'''(x) = -\cos(x) & f'''(\pi/3) = -1/2
 \end{array}$$

So we have

$$T_3(x, \pi/3) = \sqrt{3}/2 + \frac{1}{2}(x - \pi/3) - \frac{\sqrt{3}}{4}(x - \pi/3)^2 - \frac{1}{12}(x - \pi/3)^3$$

S9: Applications of Taylor Series

- (a) Use a degree-five Taylor polynomial to estimate $\sin(.3)$. Give an upper bound for the error in that estimate.

Solution: We have

$$\begin{aligned}
 \sin(x) &\approx x - x^3/6 + x^5/120 \\
 \sin(.3) &\approx .3 - .3^3/6 + .3^5/120 \approx .29552.
 \end{aligned}$$

We can estimate the error in two ways. One is to use Taylor's remainder theorem. We compute

$$\begin{aligned}
 |R_6(x, 0)| &= \left| \frac{f^{(6)}(z)}{6!} x^6 \right| \\
 &= \left| \frac{-\sin(z)}{6!} x^6 \right| \\
 &\leq \frac{x^6}{6!} \\
 |R_6(.3, 0)| &\leq \frac{.3^6}{6!} \approx \frac{1.01}{10^6}.
 \end{aligned}$$

If we're clever, we could notice that while we computed T_5 , that's the same as T_6 , so

we could actually bound the error with

$$\begin{aligned} |R_7(x, 0)| &= \left| \frac{f^{(7)}(z)}{7!} x^7 \right| \\ &= \left| \frac{-\cos(z)}{7!} x^7 \right| \\ &\leq \frac{x^7}{7!} \\ |R_7(.3, 0)| &\leq \frac{.3^7}{7!} \approx \frac{4.34}{10^8}. \end{aligned}$$

But we can be much lazier if we notice that our series is an alternating series. The error, then, is bounded by the next term, which would be $.3^7/7!$. That's exactly what we got in the last computation, for an error bound of $\frac{4.34}{10^8}$.

(b) If $f(x) = \sum_{n=0}^{\infty} \frac{n+1}{n!+1} x^n$, compute $\int_3^5 f(x)$.

Solution:

$$\begin{aligned} \int f(x) &= \sum_{n=0}^{\infty} \frac{1}{n!+1} x^{n+1} + C \\ \int_3^5 f(x) &= \sum_{n=0}^{\infty} \frac{1}{n!+1} (5^{n+1} - 3^{n+1}). \end{aligned}$$

(c) Use a Taylor series to compute $\lim_{x \rightarrow 0} \frac{xe^{x^3} - x - x^4}{x^7} =$

Solution:

$$\begin{aligned} \lim_{x \rightarrow 0} \frac{xe^{x^3} - x - x^4}{x^7} &= \lim_{x \rightarrow 0} \frac{(x + x^4 + x^7/2 + x^{10}/3! + \dots) - x - x^4}{x^7} \\ &= \lim_{x \rightarrow 0} \frac{x^7/2 + x^{10}/3! + \dots}{x^7} \\ &= \lim_{x \rightarrow 0} \frac{1}{2} - \frac{x^3}{3!} + \dots = \frac{1}{2}. \end{aligned}$$