Math 1232: Single-Variable Calculus 2
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Recitation 12

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Problem 1. Consider the function $f(x) = \frac{1}{1+x^6}$.

(a) Could you compute $\int \frac{1}{1+x^6}\,dx$? How?

(b) Does it help if I tell you that $1 + x^6 = (1+x^2)(x^2 - \sqrt{3}x + 1)(x^2 + \sqrt{3}x + 1)$?

(c) Now write a power series for $f(x)$ centered at 0. What is the interval of convergence?

(d) Compute the integral of your power series. What is the interval of convergence there?

Solution:

(a) You'd have to do some sort of partial fractions thing.

(b) That helps a little, but I don't want to do that partial fractions thing and then I don't want to complete the square for those denominators.

(c)

$$\frac{1}{1+x^6} = \sum_{n=0}^{\infty} (-x^6)^n = \sum_{n=0}^{\infty} (-1)^n x^{6n}$$

This is a geometric series, so converges for $|-x^6| < 1$ and thus for $|x| < 1$. Thus the radius of convergence is 1 and the interval of convergence is $(-1, 1)$.

(d)

$$\int \frac{1}{1+x^6}\,dx = \sum_{n=0}^{\infty} \int (-1)^n x^{6n}\,dx = \sum_{n=0}^{\infty} (-1)^n \frac{x^{6n+1}}{6n+1} + C.$$

This converges for $|x| < 1$. We can see that either by using the ratio test, or by knowing that integrating or differentiating can't change the radius of c convergence.

On the endpoints, we have to use some other tool. When $x = 1$ we get the series

$$\sum_{n=0}^{\infty} (-1)^n \frac{1}{6n+1}$$

(where we can ignore the $+C$ because a constant won't affect convergence). This is an alternating series and $\lim_{n\to\infty} \frac{1}{6n+1} = 0$, so it converges by the alternating series test.

When $x = -1$ we get the series

$$\sum_{n=0}^{\infty} (-1)^n \frac{(-1)^{6n+1}}{6n+1} = \sum_{n=0}^{\infty} (-1)^n \frac{-1}{6n+1}.$$

This is again an alternating series, and $\lim_{n\to\infty} \frac{1}{6n+1} = 0$, so this series also converges.

Consequently the interval of convergence is $[-1, 1]$. Note that while the radius of convergence is different from the geometric series of part c, the interval is different—we've added in the two endpoints!

Problem 2. We want to compute $\int_3^4 \frac{1}{1-(x-4)^3}\,dx$

(a) Find a power series for to compute $\frac{1}{1-(x-4)^3}$. Where is it centered? Will it converge from 3 to 4?

(b) Find an antiderivative for this power series. Where will it converge?

(c) Now compute the integral from 3 to 4. You should get a series. Does it converge? How could we predict that?

(d) Sum the first five terms with a calculator to estimate $\int_3^4 \frac{1}{1-(x-4)^3}\,dx$.

(e) Use an online integral calculator to find the integral. How close is your answer to the true answer?

(f) What would you predict about $\int_3^5 \frac{1}{1-(x-4)^3}\,dx$?

Solution:

(a)

$$\frac{1}{1-(x-4)^3} = \sum_{n=0}^{\infty} (x-4)^{3n}$$

This is centered at 4, so it converges on $(3, 5)$. But it won't converge at 3 or at 5.

(b)

$$\begin{aligned}\int_3^4 \frac{1}{1-(x-4)^3}\,dx &= \sum_{n=0}^{\infty} \int (x-4)^{2n}\,dx \\ &= \sum_{n=0}^{\infty} \frac{(x-4)^{3n+1}}{3n+1} + C.\end{aligned}$$

This will still converge on $(3, 5)$. We can check that it diverges at 5 (by limit comparison to $1/n$) but converges at 3 (by the Alternating Series Test). So the interval of convergence is $[3, 5)$.

(c)

$$\begin{aligned}\int_3^4 \frac{1}{1-(x-4)^3}\,dx &= \sum_{n=0}^{\infty} \int (x-4)^{2n}\,dx \Big|_3^4 \\ &= \sum_{n=0}^{\infty} \frac{(x-4)^{3n+1}}{3n+1}\Big|_3^4 \\ &= 0 - \sum_{n=0}^{\infty} \frac{(-1)^{3n+1}}{3n+1}\end{aligned}$$

which converges by the Alternating Series Test.

This makes sense, because the power series converged on $[3, 5)$ so the integral should converge everywhere in that interval.

(d)

$$\sum_{n=0}^{5} \frac{(-1)^{3n+1}}{3n+1} = 1 - \frac{1}{4} + \frac{1}{7} - \frac{1}{10} + \frac{1}{13} - \frac{1}{16} = \frac{5877}{7280} \approx 0.80728.$$

(e) The true answer is about 0.835649 so this is prety decent.

(f) $\int_3^5 \frac{1}{1-(x-4)^3}\,dx$ doesn't converge. We can see from our previous calculations that the power series won't converge at 5, and so the integral doesn't converge. In fact the original integral $\int_3^5 \frac{1}{1-(x-4)^3}\,dx$ is an improper integral that doesn't converge at all.

Problem 3. Let's find the Taylor series of $f(x) = e^x$ centered at $a = 1$.

(a) Compute f', f'', f'''. Find a formula for $f^{(n)}(x)$.

(b) Give a formula for $T_f(x, 1)$.

(c) We want to know if $f(x) = T_f(x, 1)$. Find a formula for $R_k(x, 1)$. Can you show this goes to 0 as k goes to infinity?

(d) We already have another power series for f:

$$T_f(x, 0) = \sum_{n=0}^{\infty} \frac{x^n}{n!}.$$

You should have a different power series; but can you convince yourself it *should* give the same function? (What happens if you plug $x - 1$ into this series?)

Solution:

(a) $f^{(n)}(x) = e^x$.

(b) We know that $e^1 = e$, so we have

$$T_f(x, 1) = \sum_{n=0}^{\infty} \frac{e}{n!}(x - 1)^n.$$

(c) We compute that

$$R_k(x, 1) = \frac{e^z}{(k + 1)!}(x - 1)^{k+1}.$$

If $x > 0$, we know that $e^z < e^x$; if $x < 0$ then $e^z < 1$. Either way, we can fix x, and as k goes to infinity this goes to 0. So f is equal to its Taylor series centered at 1.

(d) We see that

$$T_f(x - 1, 0) = \sum_{n=0}^{\infty} \frac{1}{n!}(x - 1)^n$$

which is almost the same as $T_f(x, 1)$. This makes sense: from properties of e^x, we know that

$$e^x = e \cdot e^{x-1} = e \cdot \sum_{n=0}^{\infty} \frac{1}{n!}(x - 1)^n$$
$$= \sum_{n=0}^{\infty} \frac{e}{n!}(x - 1)^n = T_f(x, 1).$$

Problem 4. Let's do something silly, and compute the Taylor series of a polynomial.

(a) Let $f(x) = x^3 + 3x^2 + 1$. Find the Taylor series centered at zero. Was that what you expected?

(b) Now find the Taylor series centered at 2. Do you get the same thing? What's useful about this?

Solution:

(a) Let $f(x) = x^3 + 3x^2 + 1$. Then we have $f'(x) = 3x^2 + 6x, f''(x) = 6x + 6, f'''(x) = 6$, and $f^{(n)}(x) = 0$ for $n > 3$. Thus the Taylor series centered at 0 is

$$\begin{aligned} T_f(x, 0) &= f(0) + f'(0)x + \frac{f''(0)}{2}x^2 + \frac{f'''(0)}{6}x^3 \\ &= 1 + 0x + \frac{6}{2}x^2 + \frac{6}{6}x^3 = 1 + 3x^2 + x^3. \end{aligned}$$

Hopefully this is what you expected.

If we take the Taylor series centered at 2, for instance, we have

$$\begin{aligned} T_f(x, 2) &= f(2) + f'(2)x + \frac{f''(2)}{2}x^2 + \frac{f'''(2)}{6} \\ &= 21 + 24(x-2) + \frac{18}{2}(x-2)^2 + \frac{6}{6}(x-2)^3 \\ &= 21 + 24(x-2) + 9(x-2)^2 + (x-2)^3. \end{aligned}$$

If you multiply this out you will get your original polynomial back, so this is the same thing. But sometimes it is very useful to have a polynomial expressed in terms of $x - 2$, say, instead of in terms of x. This is the easiest way I know of to rewrite your polynomial that way.

Problem 5. Back in section 2 we talked about the bell curve function $p(x) = e^{-x^2}$. (Technically we should be talking about $\frac{1}{\sqrt{2\pi}}e^{-x^2/2}$ but that's annoying and doesn't change the details enough to be interesting.)

(a) Find a power series for $p(x)$ centered at zero. (This should not require any real calculations.)

(b) Find an antiderivative for $p(x)$, using power series.

(c) Write down a series that computes $\int_0^1 p(x)\,dx$.

(d) Add up the first three or four terms of this series. What do you get? Can you estimate the error in this calculation?

Solution:

(a)

$$e^{-x^2} = \sum_{n=0}^{\infty} \frac{(-x^2)^n}{n!} = \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n}}{n!}$$

(b)

$$\int e^{-x^2}\,dx = \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n+1}}{(2n+1)(n!)} + C.$$

(c)

$$\begin{aligned}\int_0^1 e^{-x^2}\,dx &= \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n+1}}{(2n+1)(n!)}\Big|_0^1 \\ &= \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)(n!)}\end{aligned}$$

(d)

$$\begin{aligned}\int_0^1 e^{-x^2}\,dx &\approx \sum_{n=0}^{3} \frac{(-1)^n}{(2n+1)(n!)} \\ &= 1 - \frac{1}{3} + \frac{1}{10} - \frac{1}{42} = \frac{26}{35} \approx 0.74.\end{aligned}$$

This is an alternating series, so the error has to be smaller than the next term $\frac{1}{9 \cdot 24} = \frac{1}{216} \approx .005$. So this is correct to two decimal places.

(In fact the true integral is 0.746824.)