Math 1232 Spring 2026
Single-Variable Calculus 2
Mastery Quiz 13
Due Thursday, April 23

This week's mastery quiz has three topics. Everyone should submit work on S10. If you have a 4/4 on M4, or a 2/2 on S9, you don't need to submit it again. This is the last quiz with S9.

Don't worry if you make a minor error, but try to demonstrate your mastery of the underlying material. Remember that you are trying to demonstrate that you understand the concepts involved. For all these problems, justify your answers and explain how you reached them. Do not just write "yes" or "no" or give a single number.

Feel free to consult your notes, but please **don't discuss the actual quiz questions with other students in the course**.

Please turn this quiz in class on Thursday. You may print this document out and write on it, or you may submit your work on separate paper; in either case make sure your name and recitation section are clearly on it.

Topics on This Quiz

- Major Topic 4: Taylor Series
- Secondary Topic 9: Applications of Taylor Series
- Secondary Topic 10: Parametrization

Name:

Recitation Section:

M4: Taylor Series

(a) Use a degree-three Taylor polynomial to estimate $(1.1)^{3.1}$.

Solution:

$$(1.1)^{3.1} \approx 1 + 3.1x + \frac{3.1 \cdot 2.1}{1 \cdot 2}x^2 + \frac{3.1 \cdot 2.1 \cdot 1.1}{1 \cdot 2 \cdot 3}x^3$$
$$= 1 + 3.1x + 3.255x^2 + 1.1935x^3$$
$$(1.1)^{3.1} \approx 1 + 3.1(.1) + 3.255(.1)^2 + 1.1935(.1)^3 = 1 + .31 + .03255 + .0011935 = 1.3437435.$$

(b) Let $f(x) = \cos^2(x)$. Use *the definition of a Taylor series* to find $T_4(x, \pi)$ for this function. (That is, find the terms up through the degree four term.)

Solution:

$$\begin{aligned} f(x) &= \cos^2(x) & f(\pi) &= 1 \\ f'(x) &= -2\cos(x)\sin(x) & f'(\pi) &= 0 \\ f''(x) &= 2\sin^2(x) - 2\cos^2(x) & f''(\pi) &= -2 \\ f'''(x) &= 4\sin(x)\cos(x) + 4\cos(x)\sin(x) & f'''(\pi) &= 0 \\ f^{(4)} &= 8\cos^2(x) - 8\sin^2(x) & f^{(4)}(\pi) &= 8. \end{aligned}$$

So we have

$$T_4(x, \pi) = 1 - (x - \pi)^2 + \frac{1}{3}(x - \pi)^4.$$

(c) Using series we already know, write down a formula for the (infinite) Taylor series for $x^3 e^{(x^5/4)}$, and then write down the first four non-zero terms of this series.

Solution:

$$e^x = \sum_{n=0}^{\infty} \frac{1}{n!}x^n$$
$$e^{x^5/4} = \sum_{n=0}^{\infty} \frac{1}{n!}(x^5/4)^n = \sum_{n=0} \frac{1}{n! \cdot 4^n}x^{5n}$$
$$x^3 e^{x^5/4} = \sum_{n=0} \frac{1}{n! \cdot 4^n}x^{5n+3}$$

The first four non-zero terms are

$$x^3 + \frac{1}{4}x^8 + \frac{1}{32}x^{13} + \frac{1}{6 \cdot 64}x^{18}.$$

(Note: this is *not* T_3 or T_4. It's T_{18}!)

S9: Applications of Taylor Series

(a) Using series, compute $\int_0^{\pi} 2x\cos(x^5)\,dx$.

Solution:

$$\begin{aligned}
\cos(x) &= \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n)!}x^{2n} \\
\cos(x^5) &= \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n)!}x^{10n} \\
2x\cos(x^5) &= \sum_{n=0}^{\infty} \frac{2(-1)^n}{(2n)!}x^{10n+1} \\
\int 2x\cos(x^5)\,dx &= \sum_{n=0}^{\infty} \frac{2(-1)^n}{(2n)!(10n+2)}x^{10n+2} + C \\
\int_0^{\pi} 2x\cos(x^5)\,dx &= \sum_{n=0}^{\infty} \frac{2(-1)^n}{(2n)!(10n+2)}\pi^{10n+2}
\end{aligned}$$

(b) Use a Taylor series to compute $\lim_{x\to 0} \frac{\cos(x^2) - 1 + x^4/2}{x^8} =$

Solution:

$$\begin{aligned}
\lim_{x\to 0} \frac{\cos(x^2) - 1 + x^4/2}{x^8} &= \lim_{x\to 0} \frac{(1 - x^4/2 + x^8/4! - x^{12}/6! + \dots) - 1 + x^4/2}{x^8} \\
&= \lim_{x\to 0} \frac{x^8/4! - x^{12}/6! + \dots}{x^8} \\
&= \lim_{x\to 0} \frac{1}{4!} - \frac{x^4}{6!} + \dots = \frac{1}{24}.
\end{aligned}$$

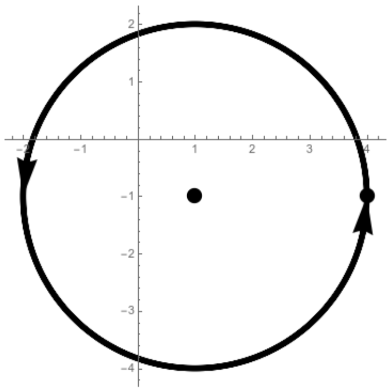
(c) Use a degree-three Taylor polynomial to estimate $(1.1)^{3.1}$.

Solution:

$$(1.1)^{3.1} \approx 1 + 3.1x + \frac{3.1 \cdot 2.1}{1 \cdot 2}x^2 + \frac{3.1 \cdot 2.1 \cdot 1.1}{1 \cdot 2 \cdot 3}x^3$$
$$= 1 + 3.1x + 3.255x^2 + 1.1935x^3$$
$$(1.1)^{3.1} \approx 1 + 3.1(.1) + 3.255(.1)^2 + 1.1935(.1)^3 = 1 + .31 + .03255 + .0011935 = 1.3437435.$$

S10: Parametrization

(a) Find a parametrization for the circle of radius 3 centered at $(1, -1)$, starting at $(4, -1)$ and going **counterclockwise twice** around the circle.



Solution: A circle has parametrization $\vec{r}(t) = (\cos(t), \sin(t))$. To make it radius 3 we multiply by 3, and then we shift it over to have center $(1, -1)$, and get

$$\vec{r}(t) = 3\cos(t) + 1, 3\sin(t) - 1.$$

In order to make it go around twice, we have $0 \leq t \leq 4\pi$. Alternatively, we could have $0 \leq 2 \leq 2\pi$ and use the equations

$$\vec{r}(t) = 3\cos(2t) + 1, 3\sin(2t) - 1.$$

(There are a bunch of other options that also work but these are the two most obvious to me.)

(b) Find the length of the curve parametrized by $x = e^t - t, y = 4e^{t/2}$ for $0 \leq t \leq 2$.

Solution: We have $x'(t) = e^t - 1$ and $y'(t) = 2e^{t/2}$, so the arc length is

$$\begin{aligned} L &= \int_0^2 \sqrt{(x'(t))^2 + (y'(t))^2}\, dt \\ &= \int_0^2 \sqrt{(e^t - 1)^2 + (2e^{t/2})^2}\, dt \\ &= \int_0^2 \sqrt{e^{2t} - 2e^2 + 1 + 4e^t}\, dt = \int_0^2 \sqrt{e^{2t} + 2e^t + 1}\, dt \\ &= \int_0^2 e^t + 1\, dt = e^t + t\Big|_0^2 = e^2 + 2 - 1 = e^2 + 1. \end{aligned}$$

(c) Find polar coordinates for the point whose Cartesian coordinates are $(-3, 3)$.

Solution: This is directly up and to the left, so the angle with the positive x-axis is $3\pi/4$. The distance is $\sqrt{(-3)^2 + 3^2} = \sqrt{18} = 3\sqrt{2}$. So the coordinates are $(3\sqrt{2}, 3\pi/4)$. (There are many other equivalent correct answers, such as $(3\sqrt{2}, 11\pi/4)$ or $(-\sqrt{18}, -\pi/4)$.)

(d) Find Cartesian coordinates for the point whose polar coordinates are $(7, 7\pi/6)$.

Solution: The coordinates are

$$\begin{aligned} x &= r\cos(\theta) = 7\cos(7\pi/6) = -7\sqrt{3}/2 \\ y &= r\sin(\theta) = 7\sin(7\pi/6) = -7/2 \end{aligned}$$

so we get $(-7\sqrt{3}/2, -7/2)$.