Math 1232 Spring 2026
Single-Variable Calculus 2
Optional Mastery Quiz 14
Due Thursday, April 30

This week's mastery quiz has two topics. If you have a 4/4 on M4, or a 2/2 on S10, you don't need to submit them again.

Don't worry if you make a minor error, but try to demonstrate your mastery of the underlying material. Remember that you are trying to demonstrate that you understand the concepts involved. For all these problems, justify your answers and explain how you reached them. Do not just write "yes" or "no" or give a single number.

Feel free to consult your notes, but please **don't discuss the actual quiz questions with other students in the course**.

Please turn this quiz in class on Thursday. You may print this document out and write on it, or you may submit your work on separate paper; in either case make sure your name and recitation section are clearly on it.

Topics on This Quiz

- Major Topic 4: Taylor Series
- Secondary Topic 10: Parametrization

Name:

Recitation Section:

M4: Taylor Series

(a) In class we computed a Taylor series for $\sin(x)$ centered at zero. Use the degree-seven Taylor polynomial to approximate $\sin(3) \approx T_7(3,0)$. (You don't need to numerically simplify this.)

Using the Taylor series remainder, find an upper bound for the error in this approximation.

Solution: We know that

$$\begin{aligned}
\sin(x) &= \sum_{n=1}^{\infty}(-1)^n \frac{x^{2n+1}}{(2n+1)!} \\
T_7(x,0) &= x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} \\
T_7(x,3) &= 3 - \frac{27}{3!} + \frac{3^5}{5!} - \frac{3^7}{7!} = 3 - \frac{37}{6} + \frac{243}{120} - \frac{2187}{5040} \\
&= 3 - \frac{9}{2} + \frac{81}{40} - \frac{243}{560} = \frac{51}{560} \approx 0.091.
\end{aligned}$$

We know that $f^{n+1}(x) = \pm\cos(x)$ or $\pm\sin(x)$ so $|f^{n+1}(z)| \leq 1$, and thus

$$\begin{aligned}
|R_n(x)| &= \left|\frac{f^{(n+1)(z)}}{(n+1)!}x^{n+1}\right| \leq \frac{x^{n+1}}{(n+1)!} \\
|R_7(x)| &\leq \frac{x^{7+1}}{(7+1)!} \\
|R_7(3)| &\leq \frac{3^8}{8!} = \frac{729}{4480} \approx 0.16.
\end{aligned}$$

It would also be okay to observe that the eighth term is zero, so we could actually compute

$$\begin{aligned}
|R_n(x)| &= \left|\frac{f^{(n+1)(z)}}{(n+1)!}x^{n+1}\right| \leq \frac{x^{n+1}}{(n+1)!} \\
|R_8(x)| &\leq \frac{x^{8+1}}{(8+1)!} \\
|R_8(3)| &\leq \frac{3^9}{9!} = \frac{243}{4480} \approx 0.054.
\end{aligned}$$

(b) Write a power series expression for $\frac{x^4}{2-4x}$ centered a 0. What is the radius of convergence?

Solution: We know that

$$\begin{aligned}
\frac{1}{1-2x} &= \sum_{n=0}^{\infty}(2x)^n \\
\frac{x^4}{2}\frac{1}{1-2x} &= \frac{x^4}{2}\sum_{n=0}^{\infty}(2x)^n \\
&= \sum_{n=0}^{\infty}\frac{2^n}{2}x^{n+4} \\
\text{(or)} \quad &= \sum_{n=4}^{\infty}2^{n+3}x^n.
\end{aligned}$$

The radius of convergence is 1/2. We can figure that out by reasoning from the geometric series: the radius of convergence for the geometric series is 1, so it converges for $-1 < 2x < 1$ or $-1/2 < x < 1/2$. Or we can use the ratio test:

$$\lim_{n\to\infty}\left|\frac{2^{n+4}x^{n+1}}{2^{n+3}x^n}\right| = \lim_{n\to\infty}|2x|$$

and thus it converges when $2|x| < 1$.

(c) Using series we already know, write down a formula for the (infinite) Taylor series for $(1-2x)^{-3}$, and then write down the degree-four polynomial explicitly.

Solution: We can take this from the binomial series. So we have

$$\begin{aligned}
f(x) &= \sum_{n=0}^{\infty}\binom{-3}{n}(-2x)^n = \sum_{n=0}^{\infty}\binom{-3}{n}(-2)^n x^n \\
T_4(x,0) &= 1 + (-2)\frac{-3}{1}x + 4\frac{12}{2}x^2 + (-8)\frac{-60}{6}x^3 + (16)\frac{360}{24}x^4 \\
&= 1 + 6x + 24x^2 + 80x^3 + 240x^4
\end{aligned}$$

S10: Parametrization

(a) Find a (implicit, cartesian, not parametric) equation for a line tangent to the polar curve given by $r = 2 + \cos(\theta)$ at the polar coordinates $(5/2, \pi/3)$.

Solution: We have

$$\begin{aligned} x &= r\cos(\theta) = (2+\cos(\theta))\cos(\theta) \\ &= 2\cos(\theta) + \cos^2(\theta) \\ \frac{dx}{d\theta} &= -2\sin(\theta) - 2\cos(\theta)\sin(\theta) \\ x(\pi/3) &= -1 - \sqrt{3}/2 = -3\sqrt{3}/2. \end{aligned}$$

Similarly,

$$\begin{aligned} y &= r\sin(\theta) & &= (2+\cos(\theta))\sin(\theta) \\ &= 2\sin(\theta) + \sin(\theta)\cos(\theta) \\ \frac{dy}{d\theta} &= 2cos(\theta) + \cos^2(\theta) - \sin^2(\theta) \\ y'(\pi/3) &= 1 + 1/4 - 3/4 = 1/2. \end{aligned}$$

Thus

$$\frac{dy}{dx} = \frac{dy/d\theta}{dx/d\theta} \qquad = \frac{1/2}{-3\sqrt{3}/2} = \frac{-1}{3\sqrt{3}}.$$

We compute

$$\begin{aligned} x(\pi/3) &= \frac{5}{2}\cos(\pi/3) = \frac{5}{4} \\ y(\pi/3) &= \frac{5}{2}\sin(\pi/3) = \frac{5\sqrt{3}}{4} \end{aligned}$$

and thus get the equation

$$y - \frac{5\sqrt{3}}{4} = \frac{-1}{3\sqrt{3}}\left(x - \frac{5}{4}\right).$$

(b) Find a parametrization of the ellipse $x^2/4 + y^2 = 1$. (Hint: what are the x and y intercepts?)

Solution: An ellipse is like a circle, but it's wider in one direction than the other. In particular, this ellipse goes through $(2,0),(0,1),(-2,0),(0,-1)$. So we can take

$$\begin{aligned} x &= 2\cos(t) \\ y &= sin(t). \end{aligned}$$

There are also lots of other options; another would be

$$x = 2\sin(t)$$
$$y = \cos(t).$$

This would start at a different point and go clockwise instead of counterclockwise, but still cover the entire ellipse.

(c) Find the length of the curve parametrized by $x = 3t^2, y = 3t - t^3$, for $1 \leq t \leq 4$.

Solution: We have $x'(t) = 6t$ and $y'(t) = 3 - 3t^2$, so the arc length is

$$\begin{aligned}
L &= \int_1^4 \sqrt{(x'(t))^2 + (y'(t))^2}\, dt \\
&= \int_1^4 \sqrt{(6t)^2 + (3 - 3t^2)^2}\, dt \\
&= \int_1^4 \sqrt{36t^2 + 9 - 18t^2 + 9t^4}\, dt \\
&= \int_1^4 3\sqrt{1 + 2t^2 + t^4}\, dt \\
&= \int_1^4 3 + 3t^2\, dt = 3t + t^3\Big|_1^4 = 12 + 64 - 3 - 1 = 72.
\end{aligned}$$