

Math 1232 Spring 2026
Single-Variable Calculus 2
Mastery Quiz 6
Due Thursday, February 26

This week's mastery quiz has four topics. Everyone should submit work on S5. If you have a 4/4 on M2, or a 2/2 on S3 or S4, you don't need to submit them again. (Please check Blackboard to confirm your scores!) This will be the last quiz featuring S3 or S4.

Don't worry if you make a minor error, but try to demonstrate your mastery of the underlying material. Remember that you are trying to demonstrate that you understand the concepts involved. For all these problems, justify your answers and explain how you reached them. Do not just write "yes" or "no" or give a single number.

Feel free to consult your notes, but please **don't discuss the actual quiz questions with other students in the course.**

Please turn this quiz in class on Thursday. You may print this document out and write on it, or you may submit your work on separate paper; in either case make sure your name and recitation section are clearly on it.

Topics on This Quiz

- Major Topic 2: Advanced Integration Techniques
- Secondary Topic 3: Numeric Integration
- Secondary Topic 4: Improper Integrals
- Secondary Topic 5: Geometric Applications

Name:

Recitation Section:

M2: Advanced Integration Techniques

(a) $\int x^2 \sin(3x) dx =$

Solution:

$$\begin{aligned} \int x^2 \sin(3x) dx &= \frac{-1}{3} \cos(3x)x^2 - \int \frac{-1}{3} \cos(3x) \cdot 2x dx \\ &= \frac{-x^2}{3} \cos(3x) + \frac{2}{3} \int x \cos(3x) dx \\ &= \frac{-x^2}{3} \cos(3x) + \frac{2}{3} \left(\frac{1}{3} \sin(3x)x - \int \frac{1}{3} \sin(3x) dx \right) \\ &= \frac{-x^2}{3} \cos(3x) + \frac{2x}{9} \sin(3x) + \frac{2}{27} \cos(3x) + C. \end{aligned}$$

(b) Compute $\int \frac{x^2+x-4}{(x+3)^2(x+1)} dx =$

Solution:

$$\begin{aligned} \frac{x^2+x-4}{(x+3)^2(x+1)} &= \frac{A}{x+3} + \frac{B}{(x+3)^2} + \frac{C}{x+1} \\ x^2+x-4 &= A(x+3)(x+1) + B(x+1) + C(x+3)^2 \\ 2 &= -2B \Rightarrow B = -1 \\ -4 &= 4C \Rightarrow C = -1 \\ -4 &= 3A + B + 9C = 3A - 1 - 9 \Rightarrow A = 2 \\ \frac{x^2+x-4}{(x+3)^2(x+1)} &= \frac{2}{x+3} + \frac{-1}{(x+3)^2} + \frac{-1}{x+1} \\ \int \frac{x^2+x-4}{(x+3)^2(x+1)} dx &= \int \frac{2}{x+3} + \frac{-1}{(x+3)^2} + \frac{-1}{x+1} dx \\ &= 2 \ln|x+3| + \frac{1}{x+3} - \ln|x+1| + C. \end{aligned}$$

(c) $\int (x-1)^3 \sqrt{2x-x^2} dx =$

Solution: We're going to set $x - 1 = \sin \theta$ so that $dx = \cos \theta d\theta$. Then

$$\begin{aligned} \int (x-1)^3 \sqrt{2x-x^2} dx &= \int (x-1)^3 \sqrt{1-(x-1)^2} dx \\ &= \int \sin^3(\theta) \sqrt{1-\sin^2(\theta)} \cos(\theta) d\theta \\ &= \int \sin^3(\theta) \cos^2(\theta) d\theta \\ &= \int \sin(\theta) \cos^2(\theta) - \sin(\theta) \cos^4(\theta) d\theta \\ &= -\frac{1}{3} \cos^3(\theta) + \frac{1}{5} \cos^5(\theta) + C. \end{aligned}$$

But

$$\cos(\theta) = \cos(\arcsin(x-1)) = \sqrt{1-(x-1)^2} = \sqrt{2x-x^2},$$

so

$$\int (x-1)^3 \sqrt{2x-x^2} dx = \frac{1}{5} \sqrt{2x-x^2}^5 - \frac{1}{3} \sqrt{2x-x^2}^3 + C.$$

S3: Numeric Integration

- (a) Let $f(x) = x^3 + x$. How many intervals do you need with the midpoint rule to approximate $\int_1^2 x^3 + x dx$ to within $1/10$? Compute the integral with that many intervals. (Feel free to use a calculator to plug values into f , but show every step.)

Solution: We have

$$\begin{aligned} f''(x) &= 6x \\ f'(2) &= 12 \\ |E_M| &\leq \frac{12 \cdot 1^3}{24 \cdot n^2} \leq \frac{1}{10} \\ n^2 &\geq 5 \\ n &> 2 \end{aligned}$$

so we need to use at least three intervals. Then the midpoint approximation would be

$$\int_1^2 x^3 + x dx \approx \frac{1}{3}f(7/6) + \frac{1}{3}f(9/6) + \frac{1}{3}f(11/6) \approx \frac{1}{3}(2.75 + 4.875 + 8.00) = \frac{1}{3}15.625 \approx 5.21.$$

(Since the true answer is 5.25 this is in fact within our error bound.)

(b) Suppose we have

$$g(0) = 2.4 \quad g(1) = 4 \quad g(2) = 2.7 \quad g(3) = 2.3 \quad g(4) = 1.7$$

Approximate $\int_0^4 g(x) dx$ using the Trapezoid rule, and then using Simpson's rule.

Solution: For the trapezoid rule, we have

$$\begin{aligned} T_4 &= 1 \cdot \frac{2.4 + 4}{2} + 1 \cdot \frac{4 + 2.7}{2} + 1 \cdot \frac{2.7 + 2.3}{2} + 1 \cdot \frac{2.3 + 1.7}{2} \\ &= \frac{1}{2}(6.4 + 6.7 + 5 + 4.0) = \frac{1}{2} \cdot 22.1 = 11.05. \end{aligned}$$

For Simpson's rule, we have

$$\begin{aligned} S_4 &= \frac{1}{3}(2.4 + 4 \cdot 4 + 2 \cdot 2.7 + 4 \cdot 2.3 + 1.7) \\ &= \frac{1}{3}(2.4 + 12 + 5.4 + 9.2 + 1.9) = \frac{1}{3} \cdot 34.7 \approx 11.5667. \end{aligned}$$

S4: Improper Integrals

Solution:

(a) Compute $\int_1^2 \frac{dx}{x \ln(x)} =$

Solution:

$$\begin{aligned} \int_1^2 \frac{dx}{x \ln(x)} &= \lim_{t \rightarrow 1^+} \int_t^2 \frac{dx}{x \ln(x)} \\ &= \lim_{t \rightarrow 1^+} \ln(|\ln(x)|) \Big|_t^2 \\ &= \lim_{t \rightarrow 1^+} \ln(|\ln(2)|) - \ln|\ln(t)| \end{aligned}$$

But $\lim_{t \rightarrow 1^+} \ln(t) = 0$, so $\lim_{t \rightarrow 1^+} \ln|\ln(t)| = -\infty$. So this limit diverges.

(b) Compute $\int_1^\infty x e^{-x^2} dx$.

Solution: We'll take $u = -x^2$ so $du = -2x dx$. Then

$$\begin{aligned} \int_1^\infty x e^{-x^2} dx &= \lim_{t \rightarrow \infty} \int_1^t x e^{-x^2} dx \\ &= \lim_{t \rightarrow \infty} \int_{-1}^{-t^2} \frac{-1}{2} e^u du \\ &= \lim_{t \rightarrow \infty} \frac{-1}{2} e^u \Big|_{-1}^{-t^2} \\ &= \frac{-1}{2} \lim_{t \rightarrow \infty} e^{-t^2} - e^{-1} \\ &= \frac{-1}{2} (0 - e^{-1}) = \frac{1}{2e}. \end{aligned}$$

S5: Geometric Applications

(a) Compute the arc length of the curve $y = \frac{x^4}{8} + \frac{1}{4x^2}$ as x varies from 1 to 2.

Solution: We have $y' = \frac{x^3}{2} + \frac{-1}{2x^3}$, and thus

$$\begin{aligned} L &= \int_1^2 \sqrt{1 + (x^3/2 - x^{-3}/2)^2} dx = \int_1^2 \sqrt{1 + x^6/4 - 1/2 + x^{-6}/4} dx \\ &= \int_1^2 \sqrt{x^6/4 + 1/2 + x^{-6}/4} dx = \int_1^2 \sqrt{(x^3/2 + x^{-3}/2)^2} dx \\ &= \int_1^2 x^3/2 + x^{-3}/2 dx = \frac{x^4}{8} - \frac{1}{4x^2} \Big|_1^2 \\ &= \frac{16}{8} - \frac{1}{16} - \left(\frac{1}{8} - \frac{1}{4} \right) = \frac{33}{16}. \end{aligned}$$

(b) Compute the area of the surface obtained by taking the curve $y = x^3$ as x goes from 0 to 1 and rotating it around the x -axis.

Solution: We have $y' = 3x^2$, and so

$$\begin{aligned} L &= \int_0^1 2\pi x^3 \sqrt{1 + (3x^2)^2} dx = \int_0^1 2\pi x^3 \sqrt{1 + 9x^4} dx \\ &= \frac{\pi}{27} (1 + 9x^4)^{3/2} \Big|_0^1 = \frac{10^{3/2}\pi}{27} - \frac{\pi}{27}. \end{aligned}$$