Math 1232: Single-Variable Calculus 2
George Washington University   Spring 2026
Recitation 7

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Problem 1. Let $f(x) = x^2$. Let's find the arc length between $x = 0$ and $x = 4$.

(a) This makes a very reasonable shape. What does the graph look like?

(b) Set up an integral to compute this arc length. You need to think about the variable of integration, the bounds, and the actual function to integrate.

(c) What techniques should we use to compute this integral? Where do we get stuck?

(d) Is there another way we could have set it up?

(e) Is that integral any easier?

Solution:

(a) It's a parabola. We know parabolas.

(b) The problem setup tells us we have x going from 0 to 4. And $f'(x) = 2x$. So it makes sense to set up

$$L = \int_0^4 \sqrt{1 + (2x)^2}\, dx$$

(c) This looks a lot like a trig sub integral. We can set $2x = \tan\theta$, so $dx = \frac{1}{2}\sec^2\theta\, d\theta$. When $x = 0$ we have $\tan\theta = 0$ so $\theta = 0$, and when $x = 4$ we have $\tan\theta = 8$ so

$\theta = \arctan(8)$. This gives us

$$L = \int_0^4 \frac{1}{2}\sqrt{1+(2x)^2}\,dx = \int_0^{\arctan 8} \frac{1}{2}\sqrt{1+\tan^2(\theta)}\sec^2\theta\,d\theta$$
$$= \int_0^{\arctan 8} \frac{1}{2}\sec^3\theta\,d\theta$$

and at this point I...give up on this integral. We said that functions with odd numbers of secants and even numbers of tangents suck, and we just don't want to do them. But we can look it up, or plug it into a computer algebra package. We get

$$\frac{1}{4}\sec(x)\tan(x) + \frac{1}{2}\ln|\sec(x)+\tan(x)|\Big|_0^{\arctan 8} \approx 16.819.$$

(d) We could instead have treated the function as $x = \sqrt{y}$, with $\frac{dx}{dy} = \frac{1}{2\sqrt{y}}$. Then we get

$$L = \int_0^{16} \sqrt{1+\frac{1}{4y}}\,dy.$$

(e) This integral is also technically doable, but very unpleasant and it involves both integration by parts and hyperbolic trig functions.

Problem 2. Let $f(x) = \sqrt[3]{3x}$. Take the portion of the graph where $0 \leq y \leq 2$ and rotate it around the y axis.

(a) Try to sketch a picture of what this will look like.

(b) Set up an integral to find the surface area. Again, think about the variable of integration, the bounds, and the function. Do you have multiple choices here or just one?

(c) Can you compute that integral?

Solution:

(a)

(b) Because we're rotating around the y axis, we basically have to integrate with respect to y. So we have $y = \sqrt[3]{3x}$ and thus $x = y^3/3$. Then $x' = y^2$, and we have

$$SA = \int_0^2 \frac{2\pi y^3}{3}\sqrt{1+y^4}\,dy.$$

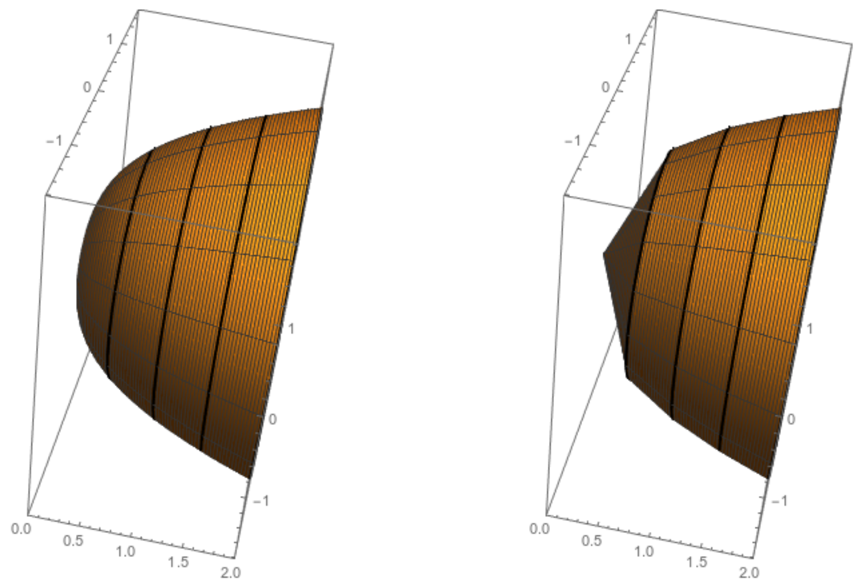

Figure 0.1: The graph of $y = \sqrt[3]{3x}$ rotated around the x-axis

(c) This one isn't so bad!

$$\begin{aligned} SA &= \int_0^2 \frac{2\pi y^3}{3} \sqrt{1+y^4}\, dy \\ &= \frac{2\pi}{3} \int_0^2 y^3 \sqrt{1+y^4}\, dy \\ &= \frac{2\pi}{3} \frac{2}{12} (1+y^4)^{3/2} \Big|_0^2 \\ &= \frac{\pi}{9} \left(17^{3/2} - 1\right) \approx 24.118. \end{aligned}$$

Problem 3. Suppose we have a Hooke's Law system of a weight on a spring. Suppose $m = k$, so that we get the differential equation $x''(t) = -x(t)$.

(a) From class, we know the general solution to this differential equation. What is it?

(b) Suppose now we start (at time 0) with the weight stationary and displaced by 1 meter. What initial conditions does this correspond to?

(c) Find the specific solution to this initial value problem.

(d) What does this describe physically? Does that solution make physical sense?

Solution:

(a) We said that then $x(t) = a\sin(t) + b\cos(t)$ for some constants a and b.

(b) These are the position and velocity at time zero, so we have $x(0) = 1$ and $x'(0) = 0$.

(c) Since $x(0) = 1$ we know that

$$1 = a\sin(0) + b\cos(0) = b,$$

and since $x'(0) = 0$ we know that

$$0 = a\cos(0) - b\sin(0) = a$$

so we have $a = 0, b = 1$, and $x(t) = \cos(t)$ is the specific solution.

(d) The weight is oscillating between plus one meter and minus one meter. This makes sense: it starts at one meter of displacement, with no velocity, so that's as far as it should ever get.

Problem 4. We can also do a different setup that is not technically an initial value problem. Suppose we have a Hooke's Law setup with a weight on a spring, and $m = k$, so $x(t) = a\sin(t) + b\cos(t)$.

(a) Suppose the weight starts with a displacement of 2, and at time $t = \pi/4$ the displacement is $\sqrt{8}$. How can we encode that mathematically?

(b) Why is this not an initial value problem? (We call this a "boundary value problem". Why do you think we call it that?)

(c) Is this enough information to find a specific solution to the differential equation? What is it?

Solution:

(a) We can write these as $x(0) = 2$ and $x(\pi/4) = \sqrt{8}$.

(b) This isn't an "initial" value problem because not all our information is about how things start. Instead we know how they start and, putatively, how they end; we know about the conditions on the boundary of some time interval.

(c)

$$\begin{aligned}
a\sin(0) + b\cos(0) &= 2 \\
b &= 2 \\
a\sin(\pi/4) + b\cos(\pi/4) &= \sqrt{8} \\
a\sqrt{2}/2 + 2\cdot\sqrt{2}/2 &= \sqrt{8} \\
a/2 + 1 &= 2 \\
a &= 2.
\end{aligned}$$

Thus we have that $x(t) = 2\sin(t) + 2\cos(t)$.