Math 2233 Summer 2026
Multivariable Calculus
Mastery Quiz 11 Worksheet
Due Wednesday, August 5

This week's mastery quiz has three topics. This is the first and only quiz featuring M6.

Remember that you are trying to demonstrate that you understand the concepts involved. For all these problems, you should justify your answers and show your work. Do not just write "yes" or "no" or give a single number.

On Wednesday, we will end class with a short quiz, containing questions drawn from this worksheet. You may not use or consult any materials or notes during Wednesday's quiz; make sure you work through these problems carefully, and understand them well enough that you can replicate that work in class.

Topics on This Quiz

- Major Topic 5: Line Integrals
- Major Topic 6: Surface Integrals
- Secondary Topic 5: Vector Fields

Name:

Name:

M5: Line Integrals

(a) Compute the arc length of the curve $\vec{r}(t) = (t^2, 3, \frac{1}{3}t^3)$ between the points $(0,3,0)$ and $(1,3,1/3)$.

Solution: We know the arc length of a curve is $\int_C 1\,ds = \int_a^b \|\vec{r}'(t)\|\,dt$. We see that our two points are $\vec{r}(0)$ and $\vec{r}(1)$, so we have

$$\begin{aligned} L &= \int_0^1 \|(2t, 0, t^2)\|\,dt \\ &= \int_0^1 \sqrt{4t^2 + t^4}\,dt = \int_0^1 t\sqrt{4 + t^2}\,dt \\ &= \frac{1}{3}(4+t^2)^{3/2}\Big|_0^1 = \frac{1}{3}(5^{3/2} - 4^{3/2}) = \frac{1}{3}(\sqrt{125} - 8). \end{aligned}$$

(b) Evaluate $\int_C \vec{G} \cdot d\vec{r}$ where $\vec{G}(x,y,z) = z\vec{i} - y\vec{j} + 2x\vec{k}$ and C is the straight line path from $(0,0,0)$ to $(1,2,3)$.

Solution: We can parametrize C with $\vec{r}(t) = (t, 2t, 3t)$ for $0 \le t \le 1$. Then the integral is

$$\begin{aligned} \int_C \vec{G} \cdot d\vec{r} &= \int_0^1 \left(3t\vec{i} - 2t\vec{j} + 2t\vec{k}\right) \cdot \left(\vec{i} + 2\vec{j} + 3\vec{k}\right) dt \\ &= \int_0^1 3t - 4t + 6t\,dt = \int_0^1 5t\,dt \\ &= \frac{5}{2}t^2 \mid_0^1 = \frac{5}{2}. \end{aligned}$$

(c) Compute $\int_C \vec{F} \cdot d\vec{r}$ where $\vec{F}(x,y,z) = (x^2 - y, y^2 + x, 1)$ and C is the intersection of $x^2 + y^2 = 1$ and $z = y^2$, **oriented counterclockwise when** viewed from the point $(0,0,100)$.

Solution: Instead of parametrizing the curve, we can just find the surface whose boundary is that curve. We can take $\vec{r}(s,t) = (s, t, t^2)$ for $s^2 + t^2 \le 1$, and then we get normal vector

$$\begin{aligned} \vec{r}_s \times \vec{r}_t = (1,0,0) \times (0,1,2t) &= \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ 1 & 0 & 0 \\ 0 & 1 & 2t \end{vmatrix} \\ &= -2t\vec{j} + \vec{k}. \end{aligned}$$

Since we want the curve to be oriented counterclockwise when viewed from above, that means we want the normal vector pointing upwards, which this one does, so this is the correct orientation.

We compute

$$\begin{aligned}\nabla \times \vec{F} &= \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ x^2 - y & y^2 + x & 1 \end{vmatrix} \\ &= 0\vec{i} + 0\vec{j} + (1+1)\vec{k} = 2\vec{k}.\end{aligned}$$

Thus we have

$$\begin{aligned}\int_C \vec{F} \cdot d\vec{S} &= \int_S \nabla \times \vec{F} \cdot \vec{S} \\ &= \int_{-1}^{1} \int_{-\sqrt{1-s^2}}^{\sqrt{1-s^2}} (2\vec{k}) \cdot (-2t\vec{k} + \vec{k})\, dt\, ds \\ &= \int_{-1}^{1} \int_{-\sqrt{1-s^2}}^{\sqrt{1-s^2}} 2\, dt\, ds.\end{aligned}$$

At this point we could convert this integral to polar coordinates, but we can also observe that we're just integrating the number 2 over the unit circle, which has area π. So the total integral here is 2π.

M6: Surface Integrals

(a) Find the surface area of the part of the paraboloid $z = 16 - x^2 - y^2$ which lies above the xy-plane.

Solution: We want to compute $\int_S 1\, dS = \iint_R \|\vec{r}_s \times \vec{r}_t\|\, ds\, dt$. We know from class that if our surface is the graph of a function, then

$$\|\vec{r}_s \times \vec{r}_t\| = \sqrt{1 + (f_x)^2 + (f_y)^2} = \sqrt{1 + 4x^2 + 4y^2}.$$

We're integrating over the circle $x^2 + y^2 = 16$, so we should use polar coordinates. So we get the integral

$$\begin{aligned}\int_0^4 \int_0^{2\pi} \sqrt{1+4r^2}\, r\, d\theta\, dr &= \int_0^4 2\pi r \sqrt{1+4r^2}\, dr \\ &= \frac{\pi}{6}(1+4r^2)^{3/2}\Big|_0^4 = \frac{\pi}{6}(65^{3/2} - 1).\end{aligned}$$

(b) Compute $\iint_S \nabla \times \vec{F} \cdot d\vec{S}$ where $\vec{F}(x,y,z) = (z-y, x, -x)$ and S is the hemisphere $x^2+y^2+z^2=4, z \geq 0$, oriented inwards towards the center of the hemisphere.

Solution: Instead of parametrizing the spehre, we can just parametrize the boundary, which is $x^2+y^2=4$ and parametrized by $(2\cos(t), 2\sin(t))$. However, we see this is the wrong orientation, so we instead use $\vec{r}(t) = (2\cos(t), -2\sin(t))$. (We could also take $(2\sin(t), 2\cos(t))$, or various other options.)

Thus we compute

$$\begin{aligned}
\iint_S \nabla \times \vec{F} \cdot d\vec{S} &= \int_C \vec{F} \cdot d\vec{R} \\
&= \int_0^{2\pi} (0 + 2\sin(t), 2\cos(t), -2\cos(t)) \cdot (-2\sin(t), -2\cos(t), 0)\, dt \\
&= \int_0^{2\pi} -4\sin^2(t) - 4\cos^2(t)\, dt \\
&= \int_0^{2\pi} -4\, dt = -8\pi.
\end{aligned}$$

(c) Compute the flux of the vector field $\vec{F}(x,y,z) = y\vec{i} + z\vec{k}$ upwards through portion of the cone $z = 1 - \sqrt{x^2+y^2}$ above the plane $z=0$.

Solution: We can parametrize the cone with $(r\cos(t), r\sin(t), 1-r)$, where $0 \leq r \leq 1$ and $0 \leq t \leq 2\pi$, and we get Jacobian term

$$\begin{aligned}
\vec{r}_r \times \vec{r}_t &= (\cos(t), \sin(t), -1) \times (-r\sin(t), r\cos(t), 0) \\
&= \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \cos(t) & \sin(t) & -1 \\ -r\sin(t) & r\cos(t) & 0 \end{vmatrix} = r\cos(t)\vec{i} + r\sin(t)\vec{j} + (r\cos^2 t + r\sin^2 t)\vec{k} \\
&= r\cos(t)\vec{i} + r\sin(t)\vec{j} + r\vec{k}.
\end{aligned}$$

This is the right orientation since it is in fact pointed upwards for positive r.

Then we compute flux as

$$\begin{aligned}
\int_S \vec{F}\cdot d\vec{S} &= \int_0^1\int_0^{2\pi}(r\sin(t),0,1-r)\cdot(r\cos(t),r\sin(t),r)\,dt\,dr\\
&= \int_0^1\int_0^{2\pi} r^2\sin(t)\cos(t)+r-r^2\,dt\,dr\\
&\int_0^1 \frac{1}{2}r^2\sin^2(t)+rt-r^2t\Big|_0^{2\pi}\,dr\\
&\int_0^1 2\pi r-2\pi r^2\,dr = \pi r^2-\frac{2\pi}{3}r^3\Big|_0^1 = \frac{\pi}{3}.
\end{aligned}$$

S5: Vector Fields

(a) Find a potential function for $\vec{G}(x,y) = xy\vec{i} + (x+y)\vec{j})$, or prove none exists.

Solution: There are two approaches we could take here. One is to set up a system of differential equations:

$$\begin{aligned}
\frac{\partial g}{\partial x}(x,y) &= xy & \frac{\partial g}{\partial y}(x,y) &= x+y\\
g(x,y) &= \frac{1}{2}x^2y+g_1(y) & g(x,y) &= xy+\frac{1}{2}y^2+g_2(x)
\end{aligned}$$

and that's a contradiction, since xy isn't a function of y and x^2y isn't a function of x.

But the simpler approach is just to compute the curl

$$\begin{aligned}
\nabla\times G(x,y) &= \left(\frac{\partial x+y}{\partial x}-\frac{\partial xy}{\partial y}\right)\vec{k}\\
&= (1-x)\vec{k}\neq\vec{0}.
\end{aligned}$$

Since the curl isn't zero, this vector field isn't conservative, and doesn't have a potential.

(b) Find a potential function for $\vec{F}(x,y) = (x+y)\vec{i} + (x-y)\vec{j}$ or prove none exists.

Solution: If $\vec{F} = \nabla f$, we must have

$$\begin{aligned}
f(x,y) &= \frac{1}{2}x^2+xy+g(y)\\
f(x,y) &= xy-\frac{1}{2}y^2+h(x)
\end{aligned}$$

and we can satisfy this with

$$f(x,y) = x^2/2+xy-y^2/2.$$

We could also compute the curl:

$$\begin{aligned}\nabla \times F(x,y) &= \left(\frac{\partial x - y}{\partial x} - \frac{\partial x + y}{\partial y}\right)\vec{k} \\ &= (1-1)\vec{k} = \vec{0}.\end{aligned}$$

Since the vector field is defined everywhere, this is enough to tell us that it's conservative, and must have a potential field. But that doesn't give us a potential function, so we still need to solve the differential equations.

(c) Which of the following is a flow line for $\vec{F}(x,y,z) = 2x\vec{i} + z\vec{j} - z^2\vec{k}$?

(i) $\vec{r}(t) = \left(e^{2t}, \ln|t|, \frac{1}{t}\right)$

(ii) $\vec{s}(t) = \left(e^{t}, \frac{1}{t}, t\right)$

Solution: We compute

$$\begin{aligned}\vec{r}'(t) &= 2e^{2t}\vec{i} + \frac{1}{t}\vec{j} - \frac{1}{t^2}\vec{k} \\ \vec{F}(\vec{r}(t)) &= 2e^{2t}\vec{i} + \frac{1}{t}\vec{j} - \frac{1}{t^2}\vec{k}\end{aligned}$$

so $\vec{r}(t)$ is a flow line.

But we also compute

$$\begin{aligned}\vec{s}'(t) &= e^{t}\vec{i} - \frac{1}{t^2}\vec{j} + \vec{k} \\ \vec{F}(\vec{s}(t)) &= 2e^{t}\vec{i} + t\vec{j} - t^2\vec{k}\end{aligned}$$

so $\vec{s}(t)$ is not a flow line.