Math 2233 Summer 2026

Multivariable Calculus

Mastery Quiz 5 Worksheet

Due Wednesday, July 15

This week's mastery quiz has two topics. This is the first quiz featuring M3, and we will see M2 again next week as well.

Remember that you are trying to demonstrate that you understand the concepts involved. For all these problems, you should justify your answers and show your work. Do not just write "yes" or "no" or give a single number.

On Wednesday, we will end class with a short quiz, containing questions drawn from this worksheet. You may not use or consult any materials or notes during Wednesday's quiz; make sure you work through these problems carefully, and understand them well enough that you can replicate that work in class.

Topics on This Quiz

- Major Topic 2: Partial Derivatives
- Major Topic 3: Optimization

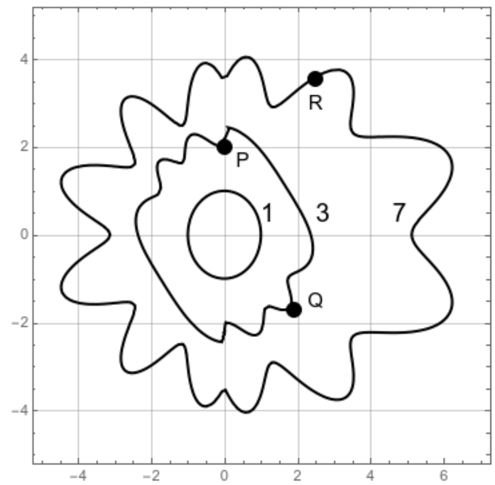
Name:

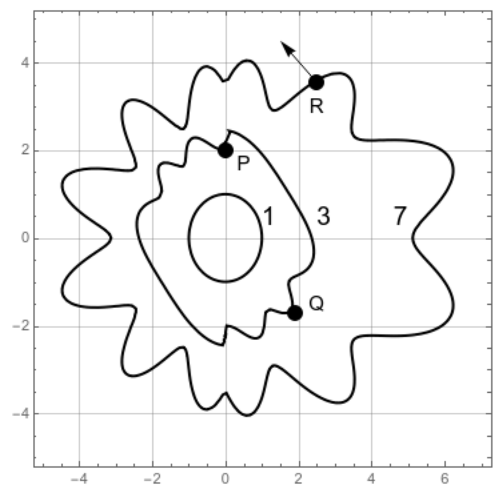
Name:

M2: Partial Derivatives

(a) Below is a contour plot of the function $h(x, y)$.

- Sketch the gradient vector at R.
- Estimate $\frac{\partial h}{\partial y}$ at the point P. Explain your reasoning in a sentence or so.





Solution:

At the point p, we see that going up by two units (one gridline, from 2 to 4) increases the output by about 4. Going down by one unit decreases the output by about 2. Consequently, the derivative should be about 2.

(b) Set $h(x, y) = ye^{2x+y}$. Use a linear approximation to estimate $h(-.9, 2.1)$.

Solution: We compute

$$\begin{aligned} h(-1,2) &= 2 \\ h_x(x,y) &= 2ye^{2x+y} \\ h_x(-1,2) &= 4 \\ h_y(x,y) &= e^{2x+y} + ye^{2x+y} \\ h_y(-1,2) &= 3 \\ h(x,y) &\approx 2 + 4(x+1) + 3(y-2) \\ h(-.9, 2.1) &\approx 2 + .4 + .3 = 2.7 \end{aligned}$$

(The exact answer is $2.8347\ldots$ so this isn't great but isn't too bad.)

(c) Let $f(x,y,z) = x^2y - yz^3$. Find the directional derivative in the direction $\vec{i} + 2\vec{j} - \vec{k}$ at the point $(1,2,1)$.

Solution: We compute

$$\begin{aligned} \nabla f(x,y,z) &= \langle 2xy, x^2 - z^3, -3yz^2 \rangle \\ \nabla f(1,2,1) &= \langle 4, 0, -6 \rangle \\ \vec{u} &= \frac{1}{\sqrt{6}}\vec{i} + \frac{2}{\sqrt{6}}\vec{j} - \frac{1}{\sqrt{6}}\vec{k} \\ f_{\vec{u}}(1,2,1) &= \nabla f(1,2,1) \cdot \vec{u} \\ &= \frac{4}{\sqrt{6}} + \frac{6}{\sqrt{6}} = \frac{10}{\sqrt{6}}. \end{aligned}$$

M3: Optimization

(a) Find (but don't classify) the critical points of $g(x,y,z) = x^2y - xz^2$.

Solution: We have

$$\begin{aligned} g_x(x,y,z) &= 2xy - z^2 \\ g_y(x,y,z) &= x^2 \\ g_z(x,y,z) &= -2xz \end{aligned}$$

The second equation says that $x = 0$. Then the first says that $z = 0$ and the third is totally redundant, so we have a critical point whenever $x = z = 0$.

(Important note: y is totally unconstrained here! So there are infinitely many critical points.)

(b) Find and classify the critical points of $f(x, y) = 2x^3 - 6xy + y^2$.

Solution: We have

$$f_x(x, y) = 6x^2 - 6y$$
$$f_y(x, y) = -6x + 2y$$

The second equation tells us $y = 3x$. Substituting that into the first equation gives $6x^2 - 18x = 0$ and thus either $x = 0$ or $x = 3$. So our two critical points are $(0, 0)$ and $(3, 9)$.

We have

$$\begin{aligned} f_{xx}(x, y) &= 12x & f_{xx}(0, 0) &= 0 & f_{xx}(3, 9) &= 36 \\ f_{xy}(x, y) &= -6 & f_{xy}(0, 0) &= -6 & f_{xy}(3, 9) &= -6 \\ f_{yy}(x, y) &= 2 & f_{yy}(0, 0) &= 2 & f_{yy}(3, 9) &= 2 \end{aligned}$$

Then for $(0, 0)$ we have $D = 0 \cdot 2 - (-6)^2 = -36 < 0$, so this is a saddle point.

For $(3, 9)$ we have $D = 36 \cdot 2 - (-6)^2 = 36 > 0$, and $f_{xx} = 36 > 0$, so this is a local minimum.

(c) Find the maximum and minimum values of $f(x, y, z) = x + y^2 + 3z$ given that $4x^2 + 8y^2 + 36z^2 \leq 36$. Justify your claim that this is the maximum and minimum.

Solution: This is a continuous function on a closed and bounded region, so there must be some maximum and some minimum. If we check all the interior critical points and all the boundary critical points, the largest of those values must be the maximum and the least must be the minimum.

We first look for interior critical points. We have $\nabla f(x, y, z) = \langle 1, 2y, 3 \rangle$. This is never zero, so we have no interior critical points. We now consider the boundary.

We have

$$
\begin{aligned}
1 &= \lambda 8x \\
2y &= \lambda 16y \\
3 &= \lambda 72z.
\end{aligned}
$$

The second equation tells us that $y = 0$ or $\lambda = 1/8$.

If $\lambda = 1/8$, then the first equation tells us that $x = 1$ and $z = 1/3$. Plugging this into the constraint equation gives us

$$
\begin{aligned}
4 + 8y^2 + 4 &= 36 \\
8y^2 &= 28 \\
y &= \pm\sqrt{7/2}.
\end{aligned}
$$

This gives us two critical points: $(1, \sqrt{7/2}, 1/3)$ and $(1, -\sqrt{7/2}, 1/3)$. We compute

$$
\begin{aligned}
f(1, \sqrt{7/2}, 1/3) &= 1 + 7/2 + 1 = 11/2 \\
f(1, -\sqrt{7/2}, 1/3) &= 1 + 7/2 + 1 = 11/2.
\end{aligned}
$$

If $y = 0$, then we have

$$
\begin{aligned}
\lambda &= \frac{1}{8x} \\
\lambda &= \frac{1}{24z} \\
24z &= 8x \\
x &= 3z
\end{aligned}
$$

and thus

$$
\begin{aligned}
36z^2 + 36z^2 &= 36 \\
z^2 &= 1/2
\end{aligned}
$$

and thus $z = \pm 1/\sqrt{2}$ and $x = \pm 3/\sqrt{2}$. Then we have two critical points, and we compute

$$
\begin{aligned}
f(3/\sqrt{2}, 0, 1/\sqrt{2}) &= 3/\sqrt{2} + 3/\sqrt{2} = 6/\sqrt{2} \\
f(-3/\sqrt{2}, 0, -1/\sqrt{2}) &= -3/\sqrt{2} - 3/\sqrt{2} = -6/\sqrt{2}
\end{aligned}
$$

So the maximum value is $11/2$ and the minimum value is $-6\sqrt{2}$.