Math 2233:Multivariable Calculus

George Washington University   Summer 2026

Recitation 9

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Problem 1. We want to integrate the function $f(x, y) = (x^2+y^2)^{-1/2}$ over the annulus with inner radius 1 and outer radius 2.

(a) Why should we use polar coordinates? There are actually two good reasons.

(b) Find the bounds of the integral.

(c) Set up and compute the integral.

Solution:

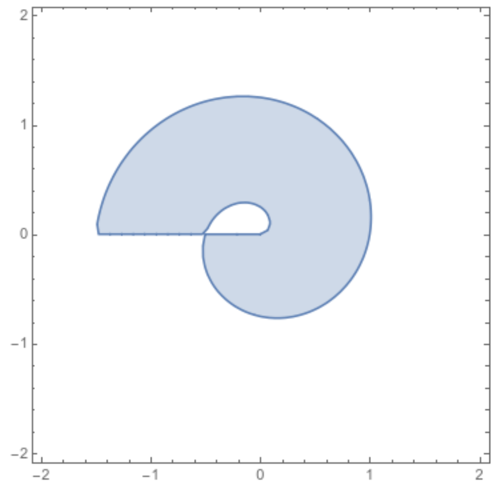
(a) The more obvious is that this region is circular, or two-circle-ular. The less obvious is that $(x^2 + y^2)^{-1/2}$ is an annoying function to integrate, but $\frac{1}{r}$ isn't bad at all.

(b) The circle goes all the way around, so $0 \leq \theta \leq 2\pi$. The inner radius is 1 and outer radius is 2, so we have $1 \leq r \leq 2$.

(c) We see that $f(x, y) = (x^2 + y^2)^{-1/2} = (r^2)^{-1/2} = \frac{1}{r}$. Thus we have

$$\begin{aligned} I &= \int_0^{2\pi} \int_1^2 \frac{1}{r} \cdot r \, dr \, d\theta = \int_0^{2\pi} r|_1^2 \, d\theta \\ &= \int_0^{2\pi} 1 \, d\theta = 2\pi. \end{aligned}$$

Problem 2. Let's find the area of the spiral that has thickness 1, and has inner radius going from 0 to 1 over one complete rotation.



(a) Should we use polar coordinates or cartesian coordinates, and why?

(b) Find a formula for the bounds of this integral. Be careful!

(c) What function should you be integrating to find the area?

(d) Compute the integral.

Solution:

(a) We want to use polar coordinates. This isn't a circle but it is *circular*, so polar should probably help.

(b) We get a full single rotation, so we can take $0 \leq \theta \leq 2\pi$. The inner radius goes from 0 to 1, so the lower bound should be $\frac{\theta}{2\pi}$. And the outer radius and outer bound should be one more than that.

(c) Because we're computing area, we just want to integrate the function 1. But because of polar coordinates our integrand needs to be r.

(d)

$$\begin{aligned}
\int_0^{2\pi} \int_{\theta/(2\pi)}^{1+\theta/(2\pi)} r\,dr\,d\theta &= \int_0^{2\pi} \left.\frac{r^2}{2}\right|_{\theta/(2\pi)}^{1+\theta/(2\pi)} d\theta \\
&= \frac{1}{2}\int_0^{2\pi} (1+\theta/(2\pi))^2 - (\theta/(2\pi))^2\,d\theta \\
&= \frac{1}{2}\int_0^{2\pi} 1 + \theta/\pi + \theta^2/(4\pi^2) - \theta^2/(4\pi^2)\,d\theta \\
&= \frac{1}{2}\int_0^{2\pi} 1 + \theta/\pi\,d\theta = \frac{1}{2}\left(\theta + \theta^2/(2\pi)\big|_0^{2\pi}\right) \\
&= \frac{1}{2}\left(2\pi + (2\pi)^2/(2\pi)\right) = 2\pi.
\end{aligned}$$

Problem 3. We want to integrate xz over wedge cut from cylinder 4 cm high and 6 cm in radius, from the positive x axis to an of angle $\pi/6$ above the x axis.

(a) Set up bounds for this integral in cylindrical coordinates.

(b) Compute the integral.

(c) (Bonus) Could you set up bounds in spherical coordinates? What would that look like?

Solution:

(a) the cylinder is 4 cm high, so we have $0 \leq h \leq 4$. The cylinder is 6 cm in radius, so we have $0 \leq r \leq 6$. And we have $0 \leq \theta \leq \pi/6$.

(b)

$$\begin{aligned}
\int_0^4 \int_0^6 \int_0^{\pi/6} r\cos(\theta) hr\,d\theta\,dr\,dh &= \int_0^4 \int_0^6 r^2 h \sin(\theta)\big|_0^{\pi/6}\,dr\,dh \\
&= \int_0^4 \int_0^6 r^2 h \frac{1}{2}\,dr\,dh \\
&= \int_0^4 \frac{r^3 h}{6}\Big|_0^6\,dh \\
&= \int_0^4 36h\,dh = 18h^2\big|_0^4 = 288.
\end{aligned}$$

(c) This works out at all, but not well. We still have $0 \leq \theta \leq 2\pi$. And it's not hard to see that we have $0 \leq \phi \leq \pi/2$, since the cylinder sits above the xy plane. But now we need to write ρ as a function of ϕ (and θ, but it looks the same at any θ).

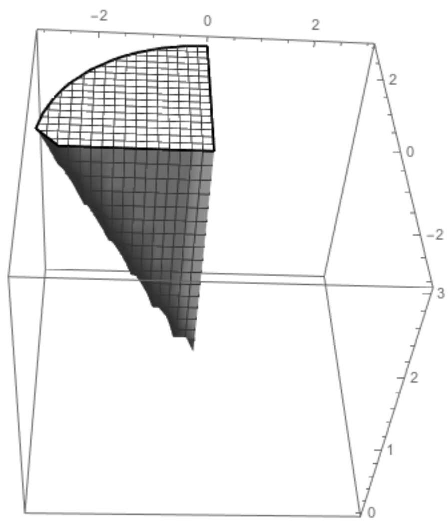
We can look at this, from the side, like a rectangle. We can draw a line from $(0,0)$ to $(6,4)$; this line is at angle $\phi = \arctan(3/2)$ from the vertical. We need to write two formulas, one for $0 \leq \phi \leq \arctan(3/2)$ and the other for $\arctan(3/2) \leq \phi \leq \pi/2$. In the upper section, we know that $\frac{4}{\rho} = \cos(\phi)$ and thus we have $\rho = \frac{4}{\cos(\phi)}$, and in the bottom section we know that $\frac{6}{\rho} = \cos(\pi/2 - \phi)$ and thus $\rho = \frac{6}{\cos(\pi/2-\phi)}$.

So we have two integrals, one with $0 \leq \theta \leq 2\pi, 0 \leq \phi \leq \arctan(3/2), 0 \leq \rho \leq \frac{4}{\cos(\phi)}$; and the other with $(0 \leq \theta \leq 2\pi, \arctan(3/2) \leq \phi \leq \pi/2, 0 \leq \rho \leq \frac{6}{\cos(\pi/2-\phi)}$. Obviously no reasonable person would do it this way.

Problem 4. Let R be the quarter-cone in the second quadrant $x \leq 0, y \geq 0$ bounded by $z = 2$, and the cone $z = \sqrt{x^2+y^2}$ (as shown below). Let $f(x,y,z) = xy$.

(a) Set up integrals to compute $\int_R f \, dA$ in cartesian, cylindrical, and spherical coordinates.

(b) Choose one of these integrals and evaluate it.



Solution:

(a) For Cartesian coordinates: the top of the cone is the circle $x^2 + y^2 = 4$, but we only have the second quadrant. So we can take $-2 \leq x \leq 0$ and then $0 \leq y \leq \sqrt{4-x^2}$, and we have $\sqrt{x^2+y^2} \leq z \leq 2$. This gives

$$\int_{-2}^{0} \int_{0}^{\sqrt{4-x^2}} \int_{\sqrt{x^2+y^2}}^{2} xy \, dz \, dy \, dx.$$

For cylindrical coordinates: We have a radius from 0 to 3, and our angle goes from $\pi/2$ to π. Then we have $r \leq z \leq 2$. We know that $x = r\cos\theta$ and $y = r\sin\theta$ so we have the integral

$$\int_0^2 \int_{\pi/2}^{\pi} \int_r^2 r^2 \sin\theta \cos\theta \cdot r \, dz \, d\theta \, dr.$$

For spherical coordinates: the side of the cone is at a 45-degree angle so we have $0 \leq \phi \leq \pi/4$. We still have $\pi/2 \leq \theta \leq \pi$. To calculate ρ we look at the cone from the side to see a right triangle; we get $\cos\phi = 2/\rho$ and thus $\rho = 2\sec\phi$. Since $x = \rho \sin(\phi)\cos(\theta)$ and $y = \rho\sin(\phi)\sin(\theta)$ we get the integral

$$\int_0^{\pi/4} \int_{\pi/2}^{\pi} \int_0^{2\sec(\phi)} \rho^2 \sin^2(\phi)\cos(\theta)\sin(\theta) \cdot \rho^2 \sin(\phi) \, d\rho \, d\theta \, d\phi.$$

(b) I really hope everyone picks the cylindrical integral. We compute

$$\begin{aligned}
I &= \int_0^2 \int_{\pi/2}^{\pi} \int_r^2 r^3 \sin\theta\cos\theta \cdot \, dz \, d\theta \, dr \\
&= \int_0^2 \int_{\pi/2}^{\pi} 2r^3 \sin(\theta)\cos(\theta) - r^4 \sin(\theta)\cos(\theta) \, d\theta \, dr \\
&= \int_0^2 r^3 \sin^2(\theta) - r^4 \sin^2(\theta)/2 \Big|_{\pi/2}^{\pi} dr \\
&= \int_0^2 r^4/2 - r^3 \, dr \\
&= r^5/10 - r^4/4 \Big|_0^2 = 16/5 - 4 = -4/5.
\end{aligned}$$