Math 2233 Summer 2026

Multivariable Calculus

Mastery Quiz 9 Worksheet

Due Wednesday, July 29

This week's mastery quiz has three topics. This is the first quiz featuring M5, and the last quiz featuring M4 or S4.

Remember that you are trying to demonstrate that you understand the concepts involved. For all these problems, you should justify your answers and show your work. Do not just write "yes" or "no" or give a single number.

On Wednesday, we will end class with a short quiz, containing questions drawn from this worksheet. You may not use or consult any materials or notes during Wednesday's quiz; make sure you work through these problems carefully, and understand them well enough that you can replicate that work in class.

Topics on This Quiz

- Major Topic 4: Integration
- Major Topic 5: Line Integrals
- Secondary Topic 4: Integral Applications

Name:

Name:

M4: Integration

(a) We want to find the volume of the region enclosed by the portion of the cylinder $x^2 + y^2 = 9$ with $y \leq 0, z \geq 0$, and the sphere $x^2 + y^2 + z^2 = 25$. Set up three different iterated integrals to compute this, in cartesian, cylindrical, and spherical coordinates. Choose one of the integrals you set up and evaluate it.

Solution: The two surfaces intersect when $9 + z^2 = 25$, so $z = \pm 4$ and we have a circle of radius 3. Thus we get

$$\int_{-3}^{3} \int_{-\sqrt{9-x^2}}^{0} \int_{0}^{\sqrt{25-x^2-y^2}} 1 \, dz \, dy \, dx.$$

I do not want to do this integral.

Cylindrical looks much better. We worked out above that the radius of the circle of intersection is 3; and theta goes from π to 2π to stay in the half-space where y is negative. Not forgetting the Jacobian we have

$$\int_{0}^{3} \int_{\pi}^{2\pi} \int_{0}^{\sqrt{25-r^2}} 1 \cdot r \, dz \, d\theta \, dr.$$

This one is very doable! We compute

$$\begin{aligned} I &= \int_0^3 \int_\pi^{2\pi} r\sqrt{25 - r^2} \, d\theta \, dr \\ &= \pi \int_0^3 r\sqrt{25 - r^2} \, dr \\ &= -\frac{\pi}{3}(25 - r^2)^{3/2}\Big|_0^3 \\ &= \frac{-\pi}{3}\left(16^{3/2} - 25^{3/2}\right) \\ &= \frac{-\pi}{3}(64 - 125) = \frac{61\pi}{3}. \end{aligned}$$

In spherical things don't work out quite so well. The equation of the top is easy, but the side wall of the cylindar is hard to handle.

We again have θ going from π to 2π, and in each direction ρ varies from 0 to 4. But we have to split our other bounds into two parts. When ϕ is between 0 and $\arctan(3/4)$, then ρ goes from 0 to 5. But then when ϕ is between $\arctan(3/4)$ and $\pi/2$, we need to do some trigonometry: we get $\rho = 3/\sin(\phi)$. Then our integral is

$$\int_{\pi}^{2\pi} \int_{0}^{\arctan(3/4)} \int_{0}^{5} 1 \cdot \rho^2 \sin\phi \, d\rho \, d\phi \, d\theta + \int_{\pi}^{2\pi} \int_{\arctan(3/4)}^{\pi/2} \int_{0}^{3/\sin(\phi)} \rho^2 \sin\phi \, d\rho \, d\phi \, d\theta$$

I don't want to do this one either.

(b) Use the change of variables $s = y, t = y - x^2$ to evaluate $\iint_R x\,dA$ over the region in the first quadrant bounded by $y = 0, y = 36, y = x^2, y = x^2 - 1$.

Solution: We have $y = s$ and $x = \sqrt{y-t} = \sqrt{s-t}$, so we compute

$$\left|\frac{\partial(x,y)}{\partial(s,t)}\right| = \begin{vmatrix} \frac{1}{2\sqrt{s-t}} & \frac{-1}{2\sqrt{s-t}} \\ 1 & 0 \end{vmatrix} = \left|\frac{1}{2\sqrt{s-t}}\right| = \frac{1}{2\sqrt{s-t}}.$$

We see that s varies from 0 to 36, and t varies from -1 to 0. The integrand is $x = \sqrt{s-t}$, so we get

$$\begin{aligned} \iint_R x\,dA &= \int_0^{36}\int_{-1}^{0} \sqrt{s-t}\frac{1}{2\sqrt{s-t}}\,dt\,ds \\ &= \int_0^{36}\int_{-1}^{0} \frac{1}{2}\,dt\,ds \\ &= \int_0^{36} \frac{1}{2}\,ds = 18. \end{aligned}$$

(c) Sketch the region of integration and compute $\iint_R xy^2\,dx\,dy$, where R is the region in the first quadrant bounded by the curves $y = x^2$ and $x = y^2$. (Do not use a calculator!)

Solution: These curves intersect at $(0,0)$ and $(1,1)$. So we can take x going from 0 to 1, and then y goes from x^2 to $\sqrt{x}$. (Coutnerintuitively, $x^2 < \sqrt{x}$ in this region.)

So we compute

$$\begin{aligned} \iint_R xy^2\,dx &= \int_0^1\int_{x^2}^{\sqrt{x}} xy^2\,dy\,dx \\ &= \int_0^1 xy^3/3\Big|_{x^2}^{\sqrt{x}}\,dx \\ &= \frac{1}{3}\int_0^1 x^{5/2} - x^7\,dx \\ &= \frac{1}{3}\left(\frac{2}{7}x^{7/2} - \frac{1}{8}x^8\right)\Big| 0^1 \\ &= \frac{1}{3}\left(2/7 - 1/8\right) = \frac{3}{56}. \end{aligned}$$

We could have set it up in the other order. Then we would get

$$\begin{aligned}
\iint_R xy^2\,dx &= \int_0^1 \int_{y^2}^{\sqrt{y}} xy^2\,dx\,dy \\
&= \int_0^1 x^2y^2/2\Big|_{y^2}^{\sqrt{y}}\,dy \\
&= \frac{1}{2}\int_0^1 y^3 - y^6\,dy \\
&= \frac{1}{2}\left(y^4/4 - y^7/7\right)\Big|_0^1 \\
&= \frac{1}{2}(1/4 - 1/7) = \frac{3}{56}.
\end{aligned}$$

I think the second way looks easier, but it's up to you! (When I worked this out, I got to the point where I was cubing $\sqrt{x}$ and realized if I did it in the other order things would work out better.)

M5: Line Integrals

(a) Compute the line integral $\int_C f(x,y)\,ds$ for $f(x,y) = xy$ where C is the first quarter of the unit circle (starting at $(1,0)$ and ending at $(0,1)$).

Solution: We have $\vec{r}(t) = (\cos(t), \sin(t))$ for $t \in [0, \pi/2]$, so

$$\begin{aligned}
\int_C f\,ds &= \int_0^{\pi/2} f(\vec{r}(t)) \cdot \|\vec{r}'(t)\|\,dt \\
&= \int_0^{\pi/2} \cos(t)\sin(t)\|\langle -\sin(t), \cos(t)\rangle\|\,dt \\
&= \int_0^{\pi/2} \cos(t)\sin(t)\sqrt{\sin^2(t) + \cos^2(t)}\,dt \\
&= \int_0^{\pi/2} \cos(t)\sin(t)\,dt \\
&= \frac{1}{2}\sin^2(t)\Big|_0^{\pi/2} = \frac{1}{2}(\sin^2(\pi/2) - \sin^2(0)) = \frac{1}{2}\cdot 1^2 = 1/2.
\end{aligned}$$

(b) Let C be the curve $y = x^2$ from $(0,0)$ to $(1,1)$.

Compute the line integral of the vector field $\vec{F}(x,y) = (xy, -x^2)$.

Solution: We parametrize the curve with $\vec{r}(t) = (t, t^2)$. Then we have

$$\int \vec{F} \cdot d\vec{r} = \int_0^1 (t^3, -t^2) \cdot (1, 2t)\, dt = \int_0^1 t^3 - 2t^3\, dt$$
$$= \int_0^1 -t^3\, dt = \frac{-t^4}{4}\Big|_0^1 = -1/4.$$

S4: Integral Applications

(a) Let R be a trapezoidal lamina bounded by the lines $y = -x/4 + 5/2, y = 0, y = 2, x = 0$, with density $\rho(x, y) = y^2$.

(i) Sketch a picture of R.

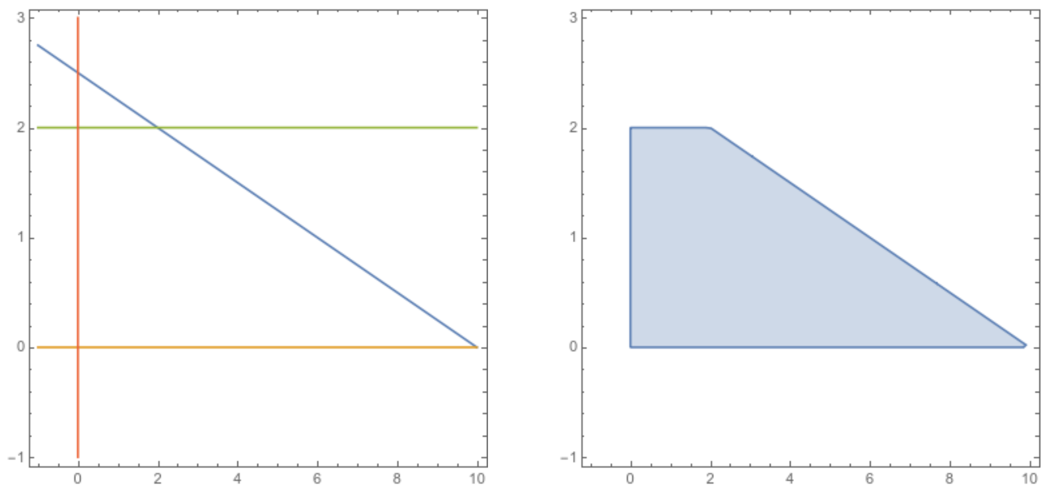
(ii) Find the mass of R.

(iii) Find the center of mass of R.

(iv) Find the moments of inertia of R.

Solution:

(i)



(ii) We want to set up an integral over the region. So we see $0 \leq y \leq 2$, and then

$0 \le x \le 10 - 4y$. So we integrate

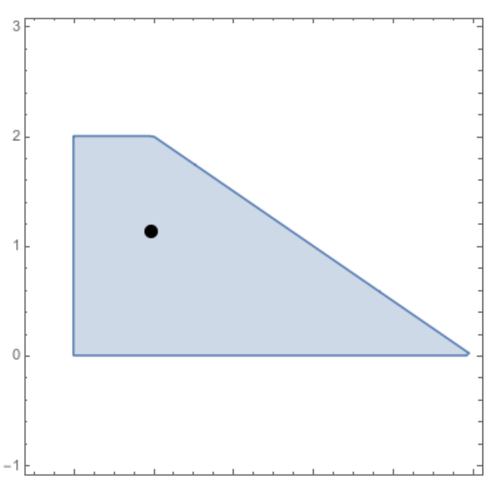
$$\begin{aligned} m &= \int_0^2 \int_0^{10-4y} y^2 \, dx \, dy \\ &= \int_0^2 10y^2 - 4y^3 \, dy \\ &= \frac{10}{3} y^3 - y^4 \Big|_0^2 = \frac{80}{3} - 16 = \frac{32}{3}. \end{aligned}$$

(iii) First we need to compute the first moments.

$$\begin{aligned} M_x &= \int_0^2 \int_0^{10-4y} y \cdot \rho(x,y) \, dx \, dy = \int_0^2 \int_0^{10-4y} y^3 \, dx \, dy \\ &= \int_0^2 10y^3 - 4y^4 \, dy = \frac{5}{2} y^4 - \frac{4}{5} y^5 \Big|_0^2 \\ &= 40 - \frac{128}{5} = \frac{72}{5}. \\ M_y &= \int_0^2 \int_0^{10-4y} x \cdot \rho(x,y) \, dx \, dy = \int_0^2 \int_0^{10-4y} xy^2 \, dx \, dy \\ &= \int_0^2 \frac{1}{2} x^2 y^2 \Big|_0^{10-4y} dy = \int_0^2 (50 - 40y + 8y^2) y^2 \, dy \\ &= \int_0^2 50y^2 - 40y^3 + 8y^4 = \frac{50}{3} y^3 - 10y^4 + \frac{8}{5} y^5 \Big|_0^2 \\ &= \frac{400}{3} - 160 + \frac{256}{5} = \frac{368}{15}. \end{aligned}$$

Thus the coordinates of the center of mass are

$$\begin{aligned} \overline{x} &= \frac{M_y}{m} = \frac{368/15}{32/3} = \frac{23}{10} \\ \overline{y} &= \frac{M_x}{m} = \frac{72/5}{32/3} = \frac{27}{20} \end{aligned}$$



(iv)

$$
\begin{aligned}
I_x &= \int_0^2 \int_0^{10-4y} y^2 \cdot \rho(x,y)\,dx\,dy = \int_0^2 \int_0^{10-4y} y^4\,dx\,dy \\
&= \int_0^2 10y^4 - 4y^5\,dy = 2y^5 - \frac{2}{3}y^6\Big|_0^2 \\
&= 64 - \frac{128}{3} = \frac{64}{3}. \\
I_x &= \int_0^2 \int_0^{10-4y} x^2 \cdot \rho(x,y)\,dx\,dy = \int_0^2 \int_0^{10-4y} x^2y^2\,dx\,dy \\
&= \int_0^2 \frac{1}{3}x^3y^2\Big|_0^{10-4y}\,dy \int_0^2 \frac{1}{3}(10-4y)^3y^2\,dy \\
&= \int_0^2 \frac{1000}{3}y^2 - 400y^3 + 160y^4 - \frac{64}{3}y^5\,dy \\
&= \frac{1000}{9}y^3 - 100y^4 + 32y^5 - \frac{32}{9}y^6\Big|_0^2 \\
&= \frac{8000}{9} - 1600 + 1024 - \frac{2048}{9} = \frac{256}{3}.
\end{aligned}
$$

(b) Set up (but do not evaluate) an integral to compute the mass of the region inside the cylinder $y^2 + z^2 = 1$ between the planes $x + y = 3$ and $x - z = 7$, if the density is $\delta(xyz) = e^{-x-yz}$.

Solution: We can take y from -1 to 1, and then z from $-\sqrt{1-y^2}$ to $\sqrt{1-y^2}$. So we just need to figure out the bounds on x. But we have $x = 7 + z$ on one side and $x = 3 - y$ on the other, so we get the integral

$$\int_{-1}^{1} \int_{-\sqrt{1-y^2}}^{\sqrt{1-y^2}} \int_{3-y}^{7+z} e^{-x-yz}\,dx\,dz\,dy.$$