Towards a Classification of 3×3 C-Symmetric Matrices

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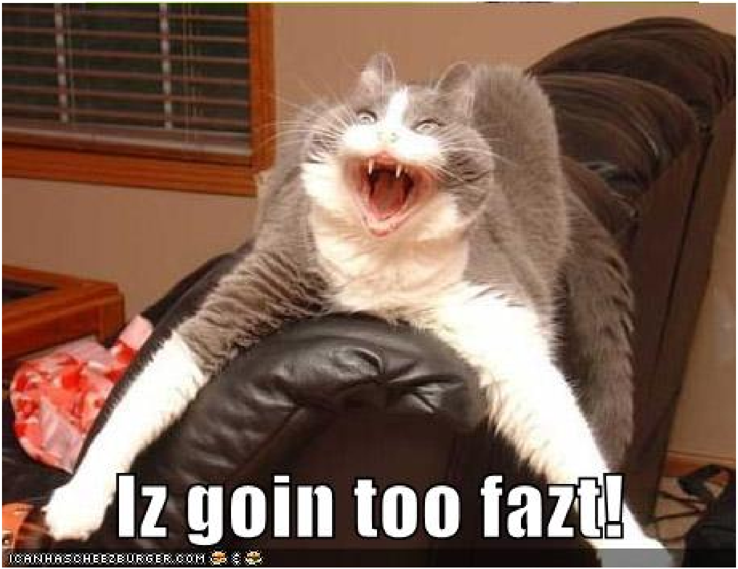
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Motivating Examples

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Fact

Every matrix is similar to a complex symmetric matrix.

Unitary Equivalence

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If A and B are $n \times n$ matrices and $A = UBU^{-1}$ for some unitary matrix U, then A is unitarily equivalent to B.

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- Develop techniques to tell the difference.
- Classify 3×3 UECSM.

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Most useful invariants are similarity invariants.

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$$\begin{aligned} \det(A) &= \det(Q^{-1}BQ) \\ &= \det(Q^{-1})\det(B)\det(Q) \\ &= \det(Q^{-1})\det(Q)\det(B) \\ &= \det(I)\det(B) = \det(B) \end{aligned}$$

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- Determinant
- Trace
- Eigenvalues
- Rank
- Minimum Polynomial
- Jordan Form

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Definition

The standard conjugation J takes a vector to its conjugate:

$$J(x_1, x_2, \ldots, x_n) = (\overline{x_1}, \overline{x_2}, \ldots, \overline{x_n}).$$

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Theorem

A matrix is UECSM if and only if it is C-symmetric for some conjugation C.

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- Direct sum of UECSM is UECSM.

A Brief Review of Eigenvectors

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Let T, v, λ be as above. If there exists a natural number n such that $(T - \lambda I)^n v = 0$ then v is a generalized eigenvector of T with eigenvalue λ.

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Thus Cv is an eigenvector of T^* with eigenvalue $\overline{\lambda}$.

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Thus C must take eigenvectors of T to corresponding eigenvectors of T^*.

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$$\begin{pmatrix} \lambda_1 & a & b \\ 0 & \lambda_2 & c \\ 0 & 0 & \lambda_3 \end{pmatrix}$$

A (Pseudo)-General Algorithm

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$|b + 0 + 0| = |0 + 0 + 0|$

$u_0 \to v_0, \; u_1 \to v_1, \; u_\lambda \to v_\lambda.$

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Rank 1   Rank 1 $|a| = |c|$

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$$2 \times 2 \oplus 1 \times 1$$

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Angle Test

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Angle Test $\qquad$ Angle Test

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Partial Isometries

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Conjecture

Every 3×3 UECSM is a rank 1 matrix, a $2 \times 2 \oplus 1 \times 1$, or some multiple of a partial isometry, plus some multiple of the identity matrix.


damn anti-gravity cat
always disproving ma theorem
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$$
\langle 2*u111*u211+2*u112*u212+2*u121*u221+2*u122*u222+2*u131*u231+2*u132*u232, 2*u111*u311+
2*u112*u312+2*u121*u321+2*u122*u322+2*u131*u331+2*u132*u332, 2*u211*u311+2*u212*u312+2*
u221*u321+2*u222*u322+2*u231*u331+2*u232*u332, 2*u111*u212+2*u131*u232-2*u122*u221+2*
u121*u222-2*u112*u211-2*u132*u231, 2*u121*u322-2*u112*u311-2*u132*u331+2*u111*u312+2*
u131*u332-2*u122*u321, -2*u222*u321+2*u221*u322-2*u212*u311-2*u232*u331+2*u211*u312+
2*u231*u332, 1-u111^2-u112^2-u121^2-u122^2-u131^2-u132^2, 1-u211^2-u212^2-u221^2-u222^2-u231^2-
u232^2, 1-u311^2-u312^2-u321^2-u322^2-u331^2-u332^2, -2*s111*u111+2*s112*u112-2*s121*u211+2*
s122*u212-2*s131*u311+2*s132*u312, 2*u111*a1-2*s111*u121-2*u112*a2-2*s121*u221+2*s112*
u122+2*s122*u222-2*s131*u321+2*s132*u322+2*u121, 2*u111*b1+2*u121*c1-2*u112*b2-2*u122*
c2+2*u131*q1-2*u132*q2-2*s111*u131-2*s121*u231+2*s112*u132+2*s122*u232-2*s131*u331+2*
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u212*a2+2*u211*a1+2*s222*u222-2*s221*u221+2*s122*u122-2*s121*u121+2*s232*u322-2*s231*
u321+2*u221, 2*u211*b1-2*u212*b2-2*s121*u131-2*u232*q2+2*u231*q1-2*u222*c2+2*u221*c1-
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u331*q1-2*s131*u131+2*s232*u232-2*s231*u231+2*s132*u132-2*s331*u331+2*s332*u332+2*u321*
c1+2*u311*b1-2*u312*b2, 2*s112*u111+2*s122*u211+2*s131*u312+2*s121*u212+2*s111*u112+2*
s132*u311, -2*u122-2*u111*a2+2*s111*u122-2*u112*a1+2*s131*u322+2*s132*u321+2*s112*u121+
2*s122*u221+2*s121*u222, -2*u132*q1-2*u131*q2-2*u111*b2-2*u112*b1-2*u121*c2-2*u122*
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2*u211*b2-2*u221*c2+2*s221*u232-2*u232*q1-2*u222*c1+2*s231*u332+2*s222*u231-2*u212*b1+
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$$


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To Paraphrase Richard Feynman:

Math is like sex. Sure, it may give some practical results, but that's not why we do it.