Non-Minimal Factorization in Numerical Monoids

Jay Daigle
gerald.daigle@pomona.edu
Pomona College

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Monoids

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A monoid is a set M with a binary associative operation $*$ and an identity element, 1. That is, for all $a, b \in M$, we have

1. $a * b \in M$
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Additive: $\mathbb{M}_{n \times m}$, $\mathbb{N}_0$.

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The numerical monoid generated *by* $n_1, \ldots, n_k$*, written* $\langle n_1, \ldots, n_k \rangle$*, is the set* $\{x_1 n_1 + x_2 n_2 + \cdots + x_k n_x \mid x_i \in \mathbb{N}_0\}$.

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Every numerical monoid has a unique minimal generating set. This set is precisely the set of irreducibles of the monoid.

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The set of lengths of x, denoted $\mathcal{L}(x)$, is

$$\{L(z) \mid z \text{ is a factorization of } x\}.$$

$$\mathcal{L}(x) = \{x_1 + x_2 + \cdots + x_n \mid x_1 n_1 + x_2 n_2 + \cdots + x_k n_k = x\}.$$

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then

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The delta set of M, denoted $\Delta(M)$, is

$$\bigcup_{x \in M} \Delta(x).$$

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Thus $\mathcal{L}(x) = \{5, 6, 7, 8, 10\}$ and $\Delta(x) = \{1, 2\}$.

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Let $M = \langle m_1, m_2, \ldots, m_l \rangle$ *and let* $S = \{n_1, n_2, \ldots, n_k\}$ *be a subset of* M *with* $\{m_1, m_2, \ldots, m_l\} \subseteq S$*. Then* S *is a* ***non-minimal basis*** *for* M*.*

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Instead of factoring elements into irreducibles, we can factor them with respect to an arbitrary basis.

Definition

Let S be a basis set for M, and let $x \in M$. Then
$\mathcal{L}^S(x) = \{x_1 + x_2 + \cdots + x_k \mid x_1 n_1 + x_2 n_2 + \cdots + x_k n_k = x\}$, *and*
$\Delta^S(x) = \{L_i - L_{i-1} \mid \mathcal{L}^S(x) = \{L_1, L_2, \ldots, L_k\}, 2 \leq i \leq k\}$.

$$\Delta^S(M) = \bigcup_{x \in M} \Delta^S(x).$$

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- What happens when we introdue one additional generator?

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Proposition

Let $M = \langle n_1, n_2 \rangle$ be a primitive numerical monoid and let $S = \{n_1, n_2, n_1 + n_2\}$. Then $\Delta^S(M) = \{1, 2, \ldots, n_2 - n_1\}$.

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- If $M = \langle 5, 11 \rangle$ and $S = \{5, 11, 55\}$, $\Delta^S(M) = \{2, 4, 6\}$.

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- If $M = \langle 12, 29 \rangle$ and $S = \{12, 29, 348\}$, $\Delta^S(M) = \{1, 2, 3, 4, 5, 6, 11, 17\}$.

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Lemma

For each $m \in M$, there exists $k \in \mathbb{N}_0$ such that $\Delta^S(m) = \Delta^S(k n_1 n_2)$.

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Theorem

Let M be a primitive numerical monoid, and $\{n_1, \ldots, n_k\}$ be any generating set for M.
For all $N \geq \left\lceil \frac{n_k}{n_1} \right\rceil n_k$, if we let $S = \{m \in M \mid m \leq N\}$, then $\Delta^S(M) = \{1\}$.

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Returning to our example $M = \langle 2, 7 \rangle$, we see that if we let $S = \{2, 7, 100\}$, we get $\Delta^S(M) = \{1, 2, 3, 4, 5, 9, 14\}$.

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Theorem

For any numerical monoid M and all $n \in \mathbb{N}$, there is a finite generating set S such that $|\Delta^S(M)| > n$.

Kaplan's Theorem

Theorem

Let $M = \langle n_1, n_2, n_3 \rangle$ *be a numerical monoid with* $n_1 < n_2 < n_3$*.*
Then $\max(\Delta(M)) = \max(\Delta(k_1 n_1) \cup \Delta(k_3 n_3))$*, where*
$k_1 = \min\{k \mid kn_1 \in \langle n_2, n_3 \rangle\}$ *and* $k_3 = \min\{k \mid kn_3 \in \langle n_1, n_2 \rangle\}$*.*

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Corollary

Let $M = \langle n_1, n_2 \rangle$ *be a numerical monoid, and let* $S = \{n_1, n_2, in_1 + jn_2\}$*.*

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Corollary

Let $M = \langle n_1, n_2 \rangle$ *be a numerical monoid, and let* $S = \{n_1, n_2, in_1 + jn_2\}$. *Then*

1. *If* $j \neq 0$ $\max(\Delta^S(H)) = \max\{n_2 - n_1, i + j - 1\}$.
2. *If* $j = 0$ *and* $n_2 < s$, $\max(\Delta^S(H)) = i - 1$.
3. *If* $j = 0$ *and* $s < n_2$, $\max(\Delta^S(H)) = \max\{i - 1, [n_2/i] + \lfloor n_2/i \rfloor - n_1\}$.

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Let $M = \langle n_1, n_2 \rangle$ *be a primitive numerical monoid and* $S = \{n_1, n_2, in_1 + jn_2\}$ *with* $i < n_2$. *Then* $i + j - 1 \in \Delta^S(M)$.

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Theorem

Let M and S be as above. Then $\Delta(M) = \Delta^S(M)$ if and only if $i + j - 1 = n_2 - n_1$.

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Let $M = \langle n_1, n_2 \rangle$ *be a primitive numerical monoid and* $S = \{n_1, n_2, in_1 + jn_2\}$ *with* $i < n_2$*. Then* $i + j - 1 \in \Delta^S(M)$.

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Let M *and* S *be as above. Then* $\Delta(M) = \Delta^S(M)$ *if and only if* $i + j - 1 = n_2 - n_1$.

Theorem

Let M *and* S *be as above. Then* $|\Delta^S(M)| = 1$ *if and only if one of the following two conditions hold:*

1. $i + j - 1 = n_2 - n_1$.
2. $j = 0$ *and* $l(i + j - 1) = n_2 - n_1$ *such that* $l \leq \lceil n_2/i \rceil$.

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Let $M = \langle n_1, n_2 \rangle$ be a primitive monoid, and let $S = \{n_1, n_2, in_1 + jn_2\}$. Suppose $i + j = 2$. Then $\Delta^S(M) = [1, k]$ for some k.

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Theorem

Let $M = \langle n_1, n_2 \rangle$ *be a primitive monoid and let* $i, j \in \mathbb{N}_0$ *such that* $i + j - 1 = k(n_2 - n_1) = k\alpha$ *for some* $k > 0$*. Then if* $S = \{n_1, n_2, in_1 + jn_2\}$, $\Delta^S(M) = \{\alpha, 2\alpha, \ldots, k\alpha\}$.

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Theorem

Let n_1, n_2 be positive relatively prime integers, and let $M = \langle n_1, n_2 \rangle$. Let $i, j \in \mathbb{N}_0$, and let $S = \{n_1, n_2, in_1 + jn_2\}$. Then if $\Delta^S(M) = \{1, k\}$, $k = 2$.

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